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SOME ASPECTS OF CONTROLLABILITY FOR NON-LINEAR PARABOLIC AND HYPERBOLIC EQUATIONS

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Thesis presented to the Postgraduate Program of Mathematics , IME - UFF, as partial fulfillment of the requirements for the degree of Doctor in Mathematics.

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Ata dos trabalhos finais da Comissão Examinadora da Tese de Doutorado em Matemática apresentada por Miguel Roberto Nuñez Chávez

Aos treze dias do mês de dezembro de dois mil e dezoito, reuniram-se no auditório da Pós-Graduação em Matemática da Universidade Federal Fluminense, os membros da Comissão Examinadora constituída pelos Professores Juan Bautista Límaco Ferrel, da Universidade Federal Fluminense; Enrique Zuazua Iriondo, da Universidad Autónoma de Madrid; Max Oliveira de Souza, da Universidade Federal Fluminense; José Felipe Linares Ramirez, do Instituto Nacional de Matemática Pura e Aplicada; Fágner Dias Araruna, da Universidade Federal da Paraíba e Reginaldo Demarque da Rocha, da Universidade Federal Fluminense, sob a presidência do primeiro, para prova pública de defesa da tese intitulada **"Some aspects of controllability for non-linear parabolic and hyperbolic equations"**, apresentada pelo doutorando Miguel Roberto Nuñez Chávez. A defesa da tese atende às exigências contidas no Regulamento Específico do Curso de Doutorado em Matemática da Universidade Federal Fluminense. A tese foi elaborada sob a orientação do Professor Juan Bautista Límaco Ferrel. O doutorando Miguel Roberto Nuñez Chávez fez a exposição de seu trabalho durante 50 minutos, iniciando às 11h e concluindo às 11h50min. A seguir, respondeu as questões formuladas pelos integrantes da Comissão Examinadora. Terminada a arguição, realizou-se a reunião da Comissão Examinadora, que apresentou parecer no sentido da aprovação do doutorando Miguel Roberto Nuñez Chávez, considerando-se o trabalho apresentado e a forma com que se houve na apresentação da defesa do mesmo. Para constar, foi lavrada a presente ata, que vai assinada pela Secretária Administrativa da Coordenação de Pós-Graduação em Matemática, pelos membros da Banca Examinadora e pelo doutorando.

Niterói, 13 de dezembro de 2018.

Prof. Juan Bautista Límaco Ferrel
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Resumo

SOME ASPECTS OF CONTROLLABILITY FOR NON-LINEAR PARABOLIC AND HYPERBOLIC EQUATIONS

Miguel Roberto Nuñez Chávez

Dezembro/2018

Orientador: Juan Límaco Ferrel (UFF)

O objetivo principal desta tese é o estudo teórico do controle nas equações do tipo parabólico e hiperbólico não lineares, temos por exemplo o estudo do controle nulo de um sistema com turbulência e acoplamento do tipo boussinesq. Neste caso vamos diminuir a quantidade de controles escalares e trabalhar sobre um domínio arbitrário. Também vamos a estudar a controlabilidade local nula de um sistema parabólico no linear unidimensional. Temos por outro lado, o estudo do controle hieráquico de uma equação do tipo calor com não linearidades no termo de difução. Aquí vamos aplicar as estratégias do tipo Stackelberg-Nash para resolver o problema. Por último estudaremos a controlabilidade exata de uma equação hiperbólica com não linearidades não locais.

Palavras-chave: Controlabilidade nula, Boussinesq, Controlabilidade Hierárquica, Parabólico, Controlabilidade Exata, Hiperbólico, Desigualdade de Carleman, Desigualdade de Observabilidade.

Abstract

SOME ASPECTS OF CONTROLLABILITY FOR NON-LINEAR PARABOLIC AND HYPERBOLIC EQUATIONS

Miguel Roberto Nuñez Chávez

December/2018

Advisor: Juan Límaco Ferrel (UFF)

The main goals of this thesis is the theoretical study of control in parabolic non-linear equations, for example, we have the study of the null control of a system with turbulence and coupling of type Boussinesq. In this case we will eliminate the number of scalar controls and work on an arbitrary domain. We will also study the null local controllability of an one-dimensional nonlinear parabolic system. On the other hand, we have the study of the hierarchical control of an equation of heat type with non-linearities in the term of diffusion. Here we will apply Stackelberg-Nash strategies to solve the problem. Finally we will study the exact controllability of a hyperbolic equation with non-local non-linearities.

Key-words: Null Controllability, Boussinesq, Hierarchical Controllability, Parabolic, Exact Controllability, Hyperbolic, Carleman Inequality, Observability Inequality.

Contents

Introduction

The study of control in PDEs has been developed during the last 30 years, on the one hand contributing with great results and on the other leaving many doubts and mysteries. Among the first to work on this theme are Jacques-Louis Lions, V. M. Alekseev and others. These works were initially focused on the optimal control using calculus of variations, then they work with the exact controllability in the equation of the wave, and later the heat equation, the first concepts of controllability were defined, the HUM method , the inequality of Carleman and the inequality of observability were studied, were great contributions for the time and this gave rise to a very large field in the area of mathematics (see [1], [25], [26]). Great characters continued to contribute to the field of control among them Andrei V. Fursikov, Oleg Yu. Imanuvilov, Enrique Zuazua, Jean-Michel Coron, Enrique Fernandez Cara, Jean Pierre Puel, Sergio Guerrero, etc. The numerical aspect begins to be worked on in this field, solving more problems of parabolic type than hyperbolic, by the simple fact of using the regularizing effect in the equations of the first type (see [7], [11], [16], [17], [18], [20], [33]).

This thesis will be divided into 5 chapters:

In Chapter 1, I will show the preliminary results and notations that I will use in the rest of Chapters.

In Chapter 2, I will deal a system that models the transfer of heat in an incompressible fluid with turbulence in dimension N. The behaivor of the similar systems have been studied by authors such as O. A. Ladyzhenskaya, T. Chacón, M. Lesieur, B. E. Launder (see [6], [21], [22], [23]). These models are the generalization of the Navier-Stokes and Boussinesq systems, which results about Controllability was worked in the next papers: [12], [13], [19]. Here Fernández-Cara, Guerrero S., Imanuvilov and Puel J. P. have worked the Local Exact Controllability with

N scalar controls. In [4], [5], Guerrero S. and Carreño N. worked the Local Null Controllability with $N-1$ scalar controls. About the principal model, there is an important result in [14], here Fernández-Cara, Límaco and Menezes have proved the theoretical and numerical results about Local Null Controllability in Smagorinsky-Ladyzhenskaya systems with N scalar controls. Last, in [28] Límaco, Nina Huaman and Nuñez-Chávez have worked in Smagorinsky-Ladyzhenskaya and Boussinesq systems with $N-1$ scalar controls. Now, in this Chapter we will work a generalization of [28], where the dependence of the main non-linearity have the spatial and temporal variables. This result is proved using the Inverse Function Theorem for Banach spaces.

In Chapter 3, I will deal a parabolic equation of heat type with non-linearity in the term of diffusion in dimension 1, results about the existence and uniqueness was worked with local initial condition by Rincón, Límaco and Liu (see [29]). This Chapter is a part of the paper [15] worked by Fernández-Cara, Nina Huaman, Nuñez-Chávez and Brito Vieira, which deal of the theoretical and numerical results of the Local Null Controllability, this paper was started in 2016 and was published in 2017. In 2017, Fernández-Cara and Límaco have worked the same results in dimensions 2 and 3, but this paper is not published yet. Actually, for dimension higher than 3, this is an open problem. Previous results are proved using Inverse Function Theorem for Banach spaces in Sobolev spaces, but there is a general result using Kakutani Fixed-Point Theorem in Holder continuous spaces for any dimension. This paper was worked by Xu Liu and Xu Zhang ([27]) in 2012. The motivation in the Chapter (or [15]) is to apply numerical methods using the Newton or Quasi-Newton algorithms obtained by Inverse Function Theorem. In the case of [27] is not clear the algorithms to employ.

In Chapter 4, I will deal Stackelberg-Nash results in a similar equation of Chapter 2, where the unique difference is the non-linear term in the diffusion. The first papers about this kind of control were made by J. L. Lions and J. Díaz in [9] where they study as restriction the approximated control. Actually, there are works of Hierarchical control with restriction of type exact controllability to trajectories, for example we have papers of Fernández-Cara, Araruna F., Santos M. and Guerrero S. (see [3] and [2]) where they have worked Stackelberg-Nash strategies with restriction of type exact controllability to trajectories for a semilinear parabolic equation for dimension less than 12. In my case, I worked in dimension 1 because I use results of the Chapter 3 and the idea of the proof is based in techniques of [3] and [2]. Working in higher dimension is a complicated problem, because I will need more regularity for the initial data and the control, and this last result maybe can not be possible to reach. For a posterior paper I have thought about working same results in dimension 2.

In Chapter 5, we will deal a hyperbolic equation with nonlocal nonlinearities in dimension 1. The work in this chapter is to obtain the Exact Controllability. The principal reference is the paper of Zuazua E. (see [33]) where he proves the Observability Inequality of an interesting way, he changes the roles of the spacial and temporal variables. I will use the classical HUM (see [24]) to prove the Exact Controllability for the linearized system and I will conclude the main result with the Schauder Fixed-Point Theorem.

Chapter 1

Preliminary Results and Notations

Let $\Omega \subset \mathbb{R}^N$ be a non-empty open bounded, connected set, with boundary $\partial\Omega$ sufficiently regular and $T > 0$.

1.1 Important Results

Lemma 1.1.1 (Fursikov's Lemma)**.** *Let $\omega_0 \subset\subset \Omega$ be a non-empty open set, there exists a function $\eta^0 \in C^2(\overline{\Omega})$ satisfying $\eta^0(x) > 0, \forall x \in \Omega$, $\eta^0(x) = 0$, $\forall x \in \partial\Omega$ and $|\nabla\eta^0(x)| > 0$, $\forall x \in \Omega\backslash\overline{\omega}$.*

Proof. See [18]. □

Theorem 1.1.2 (Inverse Function Theorem)**.** *Let Y and Z be Banach spaces and let $\mathcal{A} : B_r(0) \subset Y \to Z$ be a C^1 mapping. Let us assume that the derivative $\mathcal{A}'(0) : Y \to Z$ is onto and let us set $\xi_0 = \mathcal{A}(0)$. Then there exist $\epsilon > 0$, a mapping $W : B_\epsilon(\xi_0) \subset Z \to Y$ and a constant $K > 0$ satisfying*

$$W(z) \in B_r(0) \text{ and } \mathcal{A}(W(z)) = z, \quad \forall z \in B_\epsilon(\xi_0),$$

$$\|W(z)\|_Y \leqslant K\|z - \xi_0\|_Z, \quad \forall z \in B_\epsilon(\xi_0).$$

Proof. See [1]. □

Theorem 1.1.3 (Schauder Fixed-Point Theorem)**.** *Let Z be a Banach space and $K : Z \to Z$ a function. If exists B a compact and convex subset of Z such that $K : B \to B$ is continuous, then K admits a fixed-point in B, this is, $K(b) = b$ with $b \in B$.*

Proof. See [32]. □

1.2 Liusternik method

In Chapters 2, 3 and 4, I will follow some ideas from Fursikov and Imanuvilov [18] (see also [8]). This is:

Let be the following system

$$\begin{cases} \text{Parabolic PDE(State)} = \text{Local Control} \ \text{ in } \Omega \times (0,T), \\ \text{State} = 0 \ \text{ on } \partial\Omega \times (0,T), \\ \text{State}(0) = \text{Initial Condition} \quad \text{in } \Omega. \end{cases} \tag{1.1}$$

I define the mapping $\mathcal{A} : Y \to Z$ as

$$\mathcal{A}(\text{State}, \text{Control}) := (\text{Parabolic PDE(State)} - \text{Local Control}, \text{State}(0)).$$

where Y and Z are convenient Banach spaces such that $(0, \text{Initial Condition}) \in Z$ and $(\text{State}, \text{Local Control}) \in Y$ implies that $\text{State}(T) = 0$.

I will prove that $\mathcal{A}$ is C^1 and

$$\mathcal{A}'(0,0) = (\text{Linearized System of (1.1)(State)} - \text{Local Control}, \text{State}(0)).$$

The following result is known: $\mathcal{A}'(0,0)$ is onto is equivalent to the Null Controllability of the Linearized System of (1.1).

The steps to complete the proof are:

First, I will prove the Null Controllability of the Linearized System of (1.1) and I will prove the additional estimates to the State and Local Control.

Second, I will define the Banach spaces Y and Z and I will prove that $\mathcal{A}$ is C^1 using information of the first step.

Finally using the Theorem 1.1.2, I will conclude the Local Null Controllability to the System (1.1).

1.3 Notations

In Chapter 2, I will use the following notations:

- $Q := \Omega \times (0,T)$.
- $\partial\Omega$ is the boundary of Ω sufficiently regular.

- $\Sigma := \partial\Omega \times (0,T)$ is the lateral boundary of Q.
- $\omega \subset \Omega$ is an open non-empty.
- 1_ω denotes the characteristic function of ω.
- $Dy := \nabla y + \nabla^T y$ is the symmetric gradient of y.
- $\|\cdot\|$ and $(\cdot,\cdot)$ are the usual norm and scalar product in $L^2(\Omega)$ respectively.
- $H = \{w \in L^2(\Omega)^N : \nabla \cdot w = 0 \text{ in } \Omega, \quad w \cdot \eta = 0 \text{ on } \partial\Omega\}$.
- $V = \{w \in H_0^1(\Omega)^N : \nabla \cdot w = 0 \text{ in } \Omega\}$.
- $A : D(A) \mapsto H$ is the Stokes operator with $D(A) = H^2(\Omega)^N \cap V$, defined by $Aw = P(-\Delta w) \ \ \forall w \in D(A)$, where $P : L^2(\Omega)^N \mapsto H$ is the usual orthogonal projector.

In Chapters 3, 4 and 5, I will use the following notations:

- $I := (0, L) \subset \mathbb{R}$ is a bounded open interval.
- $\omega := (l_1, l_2) \subset I$ is an open non-empty.
- 1_ω denotes the characteristic function of ω.
- $Q := I \times (0,T)$.
- $\|\cdot\|$ and $(\cdot,\cdot)$ are the usual norm and scalar product in $L^2(I)$ respectively.
- $D_i F(s_1, s_2) := \dfrac{\partial F}{\partial s_i}(s_1, s_2)$ and $D_{ij}^2 F(s_1, s_2) := \dfrac{\partial^2 F}{\partial s_i \partial s_j}(s_1, s_2)$, for any function $F(\cdot,\cdot)$ and for any $(s_1, s_2) \in \mathbb{R}^2$.

Chapter 2

Local Null Controllabilty for Ladyzhenskaya-Smagorinsky-Boussinesq system with N-1 scalar controls

2.1 Introduction

Let $N = 2, 3$, we deal with the null controllability of the nonlinear system

$$\begin{cases} y_t - \nabla \cdot ((\nu_0 + \nu_1(\|Dy\|^2, x, t))Dy) + (y \cdot \nabla)y + \nabla p = v1_\omega + \theta e_N \ \text{ in } Q, \\ \nabla \cdot y = 0 \ \text{ in } Q, \\ \theta_t - \nabla \cdot ((\nu_0 + \nu_1(\|Dy\|^2, x, t))\nabla\theta) + y \cdot \nabla\theta = v_0 1_\omega \ \text{ in } Q, \\ y = 0, \ \ \theta = 0 \quad \text{on } \ \Sigma, \\ y(0) = y_0, \ \ \theta(0) = \theta_0 \quad \text{in } \Omega. \end{cases} \tag{2.1}$$

Here, $y = y(x,t)$, $p = p(x,t)$ and $\theta = \theta(x,t)$ represent the averaged velocity field, the pressure and the averaged temperature of a turbulent fluid respectively whose particles are in Ω during the time interval $(0,T)$; y_0 is the averaged velocity at time $t = 0$; θ_0 is the averaged temperature at time $t = 0$.

$\nu_0 \in \mathbb{R}^+$ (the kinematic viscosity of the fluid); $\nu_1 : \mathbb{R} \times \Omega \times [0,T] \to \mathbb{R}$ (the turbulent viscosity of the fluid) with $\nu_1(\cdot, x, t), \nabla\nu_1(\cdot, x, t) \in C^1(\mathbb{R})$, such that $0 \leqslant \nu_1 \leqslant C$ and $|\nabla\nu_1| + |\partial_1\nu_1| + |\nabla\partial_1\nu_1| \leqslant C$, where $\partial_1\nu_1(s,x,t) := \frac{\partial}{\partial s}\nu_1(s,x,t)$ and $\nabla\nu_1(0,x,t) = 0$

satisfying $\nu_1(0, x, \cdot) := \nu_1(0, \cdot) \in C^2([0, T])$ with $|\nu_1(0, \cdot)| + |\nu_1'(0, \cdot)| \leqslant C$.

On the other hand, v and v_0 must be viewed as controls (averaged forces) acting on the system.

For any $y_0 \in V$, $\theta_0 \in H_0^1(\Omega)$ and any $v \in L^2(\omega \times (0, T))^N$, $v_0 \in L^2(\omega \times (0, T))$ sufficiently small in their respective spaces, (2.1) possesses exactly one strong solution (y, p, θ), with

$$y \in L^2(0, T; D(A)) \cap C^0([0, T]; V), \quad y_t \in L^2(0, T; H) \tag{2.2}$$

and

$$\theta \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^0([0, T]; H_0^1(\Omega)), \quad \theta_t \in L^2(Q). \tag{2.3}$$

The main result of this chapter is given in the following theorem.

Theorem 2.1.1. *Let $i < N$ be a positive integer. Then, for every $T > 0$ and $\omega \subset \Omega$, there exist $\delta > 0$ such that for every $(y_0, \theta_0) \in V \times H_0^1(\Omega)$ satisfying $\|(y_0, \theta_0)\|_{V \times H_0^1(\Omega)} < \delta$ we can find controls $v = (v_1, \ldots, v_N) \in L^2(\omega \times (0, T))^N$ and $v_0 \in L^2(\omega \times (0, T))$, with $v_i \equiv 0$ and $v_N \equiv 0$, such that the corresponding solution (y, θ) to (2.1) satisfies (2.2) – (2.3) and*

$$y(T) = 0 \quad \textit{and} \quad \theta(T) = 0 \quad \textit{in} \quad \Omega. \tag{2.4}$$

Observation 2.1.2. *Notice that when $N = 2$, we only need to control the temperature equation.*

To prove Theorem 2.1.1 we follow a standard approach (see [4], [5], [14] and [28]). We first deduce a null controllability result for the following linear system:

$$\begin{cases} y_t - \nabla \cdot (\nu(t) Dy) + \nabla p = f + v 1_\omega + \theta e_N & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ \theta_t - \nabla \cdot (\nu(t) \nabla \theta) = f_0 + v_0 1_\omega & \text{in } Q, \\ y = 0, \quad \theta = 0 & \text{on } \Sigma, \\ y(0) = y_0, \quad \theta(0) = \theta_0 & \text{in } \Omega, \end{cases} \tag{2.5}$$

where $\nu(t) := \nu_0 + \nu_1(0, t)$, f and f_0 will be taken to decrease exponentially to zero in T.

The main tool to prove this null controllability result for system (2.5) is a suitable

Carleman estimate for the solutions of its adjoint system, namely,

$$\begin{cases} -\varphi_t - \nu(t)\Delta\varphi + \nabla\pi = g \ \text{ in } Q, \\ \nabla\cdot\varphi = 0 \ \text{ in } Q, \\ -\psi_t - \nu(t)\Delta\psi = g_0 + \varphi e_N \ \text{ in } Q, \\ \varphi = 0, \quad \psi = 0 \ \text{ on } \Sigma, \\ \varphi(T) = \varphi^T, \quad \psi(T) = \psi^T \ \text{ in } \Omega, \end{cases} \tag{2.6}$$

where $g \in L^2(Q)^N$, $g_0 \in L^2(Q)$, $\varphi^T \in H$ and $\psi^T \in L^2(\Omega)$.

This Chapter is organized as follows. In Section 2, we define preliminary results to prove the null controllability of (2.5). In Section 3, we deal with the null controllability of the linear system (2.5). Finally, in Section 4 we give the proof of Theorem 2.1.1.

2.2 Carleman Results

In this section we will introduce a Carleman estimate for the adjoint system (2.6). We are going to define some weight functions.

Let $\tau = \tau(t)$ be a positive function satisfying

$$\tau \in C^\infty([0,T]), \quad \tau(t) > 0, \ \forall t \in (0,T), \quad \tau(t) \leqslant \tau(T/2), \ \forall t \in [0,T],$$

$$\tau(t) = \begin{cases} t, & \text{if } t \leqslant T/4, \\ T - t, & \text{if } t \geqslant 3T/4. \end{cases}$$

Then for all $\lambda \geqslant 1$ we consider the following weight functions

$$\begin{cases} \alpha(x,t) = \dfrac{\alpha_0(x)}{\tau(t)^8} = \dfrac{e^{2\lambda\|\eta^0\|_\infty} - e^{\lambda\eta^0(x)}}{\tau(t)^8}, & \xi(x,t) = \dfrac{e^{\lambda\eta^0(x)}}{\tau(t)^8} \\ \alpha^*(t) = \max\limits_{x\in\overline{\Omega}} \alpha(x,t), & \xi^*(t) = \min\limits_{x\in\overline{\Omega}} \xi(x,t) \\ \hat{\alpha}(t) = \min\limits_{x\in\overline{\Omega}} \alpha(x,t), & \hat{\xi}(t) = \max\limits_{x\in\overline{\Omega}} \xi(x,t). \end{cases} \tag{2.7}$$

where η^0 is the function obtained in Lemma 1.1.1.

There exist $\lambda_{00} > 0$ such that for every $\lambda \geqslant \lambda_{00}$, we have

$$5 \max_{x\in\overline{\Omega}} \alpha_0(x) < 6 \min_{x\in\overline{\Omega}} \alpha_0(x). \tag{2.8}$$

Proposition 2.2.1. *Assume $N = 3$ and $\omega \subset\subset \Omega$. There exists a constant λ_0 such that for any $\lambda > \lambda_0$ there exist two constants $C(\lambda) > 0$ and $s_0(\lambda) > 0$ such that for any $j \in \{1, 2\}$, $g \in L^2(Q)^3$, $g_0 \in L^2(Q)$, $\varphi^T \in H$ and $\psi^T \in L^2(\Omega)$, the solution (φ, ψ) of (2.6) satisfies*

$$
\begin{aligned}
& s^4 \iint_Q e^{-5s\alpha^*}(\xi^*)^4|\varphi|^2 dxdt + s^5 \iint_Q e^{-5s\alpha^*}(\xi^*)^5|\psi|^2 dxdt \\
& \leqslant C\Bigg(\iint_Q e^{-3s\alpha^*}(|g|^2 + |g_0|^2)dxdt + s^7 \int_0^T \int_\omega e^{-2s\hat{\alpha}-3s\alpha^*}(\hat{\xi})^7|\varphi_j|^2 dxdt \\
& s^{12} \int_0^T \int_\omega e^{-4s\hat{\alpha}-s\alpha^*}(\hat{\xi})^{\frac{49}{4}}|\psi|^2 dxdt\Bigg)
\end{aligned}
\tag{2.9}
$$

for every $s \geqslant s_0$.

Proof. Making the following change of variables

$$
\tilde{\varphi}(x, s(t)) := \varphi(x, t), \quad \tilde{\psi}(x, s(t)) := \psi(x, t)
$$

with

$$
s(t) := \int_0^t \nu(r)dr, \quad t(s) = \int_0^s \frac{dr}{\nu(t(r))}
$$

we have the new system

$$
\begin{cases}
-\tilde{\varphi}_s - \Delta\tilde{\varphi} + \nabla\tilde{\pi} = \tilde{g} \;\; (x, s) \in \Omega \times (0, \int_0^T \nu(r)dr), \\
\nabla \cdot \tilde{\varphi} = 0 \;\; (x, s) \in \Omega \times (0, \int_0^T \nu(r)dr), \\
-\tilde{\psi}_t - \Delta\tilde{\psi} = \tilde{g}_0 + \tilde{\varphi}\tilde{e}_N \;\; (x, s) \in \Omega \times (0, \int_0^T \nu(r)dr), \\
\tilde{\varphi} = 0, \;\; \tilde{\psi} = 0 \;\; (x, s) \in \partial\Omega \times (0, \int_0^T \nu(r)dr), \\
\tilde{\varphi}(\int_0^T \nu(r)dr) = \varphi^T, \;\; \tilde{\psi}(\int_0^T \nu(r)dr) = \psi^T \;\text{ in } \Omega,
\end{cases}
\tag{2.10}
$$

where

$$
\tilde{\pi}(x, s) := \frac{1}{\nu(t)}\pi(x, t), \;\; \tilde{g}(x, s) := \frac{1}{\nu(t)}g(x, t), \;\; \tilde{g}_0(x, s) := \frac{1}{\nu(t)}g_0(x, t), \;\; \tilde{e}_N(s) := \frac{1}{\nu(t)}e_N.
$$

Then we apply Proposition 2.1 (see [4]) to the system (2.10) and returning for the original system, we have completed the proof. □

For the sake of completeness, let us also state this result for the 2-dimensional case.

Proposition 2.2.2. *Assume $N = 2$ and $\omega \subset\subset \Omega$. There exists a constant λ_0, such that for any $\lambda > \lambda_0$ there exist two constants $C(\lambda) > 0$ and $s_0(\lambda) > 0$ such that for* *any* $g \in L^2(Q)^2$, $g_0 \in L^2(Q)$, $\varphi^T \in H$ *and* $\psi^T \in L^2(\Omega)$, *the solution* (φ, ψ) *of* (2.6) *satisfies*

$$
\begin{aligned}
&s^4 \iint_Q e^{-5s\alpha^*}(\xi^*)^4|\varphi|^2 dxdt + s^5 \iint_Q e^{-5s\alpha^*}(\xi^*)^5|\psi|^2 dxdt \\
&\leqslant C\left(\iint_Q e^{-3s\alpha^*}(|g|^2+|g_0|^2)dxdt + s^{12}\int_0^T\int_\omega e^{-4s\hat{\alpha}-s\alpha^*}(\hat{\xi})^{\frac{49}{4}}|\psi|^2 dxdt\right)
\end{aligned}
\tag{2.11}
$$

for every $s \geqslant s_0$.

Proof. Analogously to the proof of Proposition 2.2.1. □

2.3 Null controllability of the Linear System

Here we are concerned with the null controllability of the system

$$
\begin{cases}
y_t - \nabla\cdot(\nu(t)Dy) + \nabla p = f + v1_\omega + \theta e_N \ \text{ in } Q,\\
\nabla\cdot y = 0 \ \text{ in } Q,\\
\theta_t - \nabla\cdot(\nu(t)\nabla\theta) = f_0 + v_0 1_\omega \ \text{ in } Q,\\
y = 0, \quad \theta = 0 \ \text{ on } \Sigma,\\
y(0) = y_0, \quad \theta(0) = \theta_0 \ \text{ in } \Omega,
\end{cases}
\tag{2.12}
$$

where $y_0 \in V$, $\theta_0 \in H_0^1(\Omega)$, $v \in L^2(\omega\times(0,T))^N$, $v_0 \in L^2(\omega\times(0,T))$, f and f_0 are in appropriate weighted spaces.

Before dealing with the null controllability of (2.12), we will deduce a Carleman inequality with weights not vanishing at $t = 0$. To this end, let us introduce the following weight functions

$$
\begin{cases}
\beta(x,t) = \dfrac{\alpha_0(x)}{l(t)^8} = \dfrac{e^{2\lambda\|\eta^0\|_\infty} - e^{\lambda\eta^0(x)}}{l(t)^8}, & \gamma(x,t) = \dfrac{e^{\lambda\eta^0(x)}}{l(t)^8},\\
\beta^*(t) = \max\limits_{x\in\overline{\Omega}} \beta(x,t), & \gamma^*(t) = \min\limits_{x\in\overline{\Omega}} \gamma(x,t),\\
\hat{\beta}(t) = \min\limits_{x\in\overline{\Omega}} \beta(x,t), & \hat{\gamma}(t) = \max\limits_{x\in\overline{\Omega}} \gamma(x,t).
\end{cases}
$$

where

$$
l(t) = \begin{cases}
\|\tau\|_\infty, & 0 \leqslant t \leqslant \frac{T}{2},\\
\tau(t), & \frac{T}{2} \leqslant t \leqslant T.
\end{cases}
$$

Lemma 2.3.1. *Assume $N = 3$ and $\omega \subset\subset \Omega$. Let s and λ be like in Proposition 2.2.1. Then, there exists a constant $C > 0$ (depending on s and λ) such that every solution (φ, ψ) of (2.6) satisfies*

$$\iint_Q e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2 dxdt + \iint_Q e^{-5s\beta^*}(\gamma^*)^5|\psi|^2 dxdt + \|\varphi(0)\|^2_{L^2(\Omega)^3} + \|\psi(0)\|^2_{L^2(\Omega)} \tag{2.13}$$
$$\leqslant C\left(\iint_Q e^{-3s\beta^*}(|g|^2 + |g_0|^2)dxdt + \int_0^T\int_\omega e^{-2s\hat{\beta}-3s\beta^*}(\hat{\gamma})^7|\varphi_j|^2 dxdt \right.$$
$$\left. \int_0^T\int_\omega e^{-4s\hat{\beta}-s\beta^*}(\hat{\gamma})^{\frac{49}{4}}|\psi|^2 dxdt\right)$$

for every $s \geqslant s_0$.

Proof. It is a consequence of Proposition 2.2.1. □

Let us also state this result for $N = 2$.

Lemma 2.3.2. *Assume $N = 2$ and $\omega \subset\subset \Omega$. Let s and λ be like in Proposition 2.2.2. Then, there exists a constant $C > 0$ (depending on s and λ) such that every solution (φ, ψ) of (2.6) satisfies*

$$\iint_Q e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2 dxdt + \iint_Q e^{-5s\beta^*}(\gamma^*)^5|\psi|^2 dxdt + \|\varphi(0)\|^2_{L^2(\Omega)^2} + \|\psi(0)\|^2_{L^2(\Omega)} \tag{2.14}$$
$$\leqslant C\left(\iint_Q e^{-3s\beta^*}(|g|^2 + |g_0|^2)dxdt + \int_0^T\int_\omega e^{-4s\hat{\beta}-s\beta^*}(\hat{\gamma})^{\frac{49}{4}}|\psi|^2 dxdt\right)$$

for every $s \geqslant s_0$.

Proof. It is a consequence of Proposition 2.2.2. □

Let us denote

$$\begin{cases} \tilde{\rho}(t) = e^{\frac{3}{2}s\beta^*(t)} & , \quad \hat{\rho}(t) = e^{\frac{3}{2}s\hat{\beta}(t)}, \\ \tilde{\eta}(t) = e^{s\hat{\beta}(t)+\frac{3}{2}s\beta^*(t)}\hat{\gamma}^{-\frac{7}{2}}(t) & , \quad \hat{\eta}(t) = e^{2s\hat{\beta}(t)+\frac{1}{2}s\beta^*(t)}\hat{\gamma}^{-\frac{49}{8}}(t), \\ \tilde{\sigma}(t) = e^{\frac{5}{2}s\beta^*(t)}(\gamma^*)^{-2}(t) & , \quad \hat{\sigma}(t) = e^{\frac{5}{2}s\beta^*(t)}(\gamma^*)^{-\frac{5}{2}}(t), \\ \zeta(t) = \hat{\rho}(t)\ l^{12}(t) & , \quad \kappa(t) = \hat{\rho}(t)\ l^{\frac{33}{2}}(t). \end{cases} \tag{2.15}$$

Proposition 2.3.3. *Assume $N \in \{2, 3\}$ with $j \neq N$, let $y_0 \in V$, $\theta_0 \in H^1_0(\Omega)$, $\tilde{\sigma} f \in L^2(Q)^N$, $\hat{\sigma} f_0 \in L^2(Q)$. Then we can find controls $v = (v_1, \ldots, v_N) \in L^2(\omega \times (0, T))^N$*

and $v_0 \in L^2(\omega \times (0,T))$ *such that the associated solution* (y,p,θ) *to* (2.12) *satisfies* $v_j \equiv v_N \equiv 0$*, with*

$$\tilde{\rho} y,\ \tilde{\eta} v 1_\omega \in L^2(Q)^N, \tilde{\rho}\theta,\ \hat{\eta} v_0 1_\omega \in L^2(Q). \tag{2.16}$$

In particular, $y(T)=0$ *and* $\theta(T)=0$*.*

Proof. See [4]. □

2.3.1 Some Additional Estimates

Proposition 2.3.4. *Let the assumptions in Proposition* 2.3.3 *be satisfied and let* (y,p,θ,v) *satisfy* (2.12) *and* (2.16)*. Then*

$$\sup_{[0,T]} \int_\Omega \zeta^2 |y|^2 dx + \iint_Q \zeta^2 |\nabla y|^2 dxdt \leqslant C \left(\|y_0\|^2 + \iint_Q \tilde{\sigma}^2 |f|^2 dxdt + \iint_Q \tilde{\rho}^2 |y|^2 dxdt \right.$$
$$\left. + \iint_Q \tilde{\rho}^2 |\theta|^2 dxdt \iint_{\omega \times (0,T)} \tilde{\eta}^2 |v|^2 dxdt \right)$$

and

$$\sup_{[0,T]} \int_\Omega \zeta^2 |\theta|^2 dx + \iint_Q \zeta^2 |\nabla \theta|^2 dxdt \leqslant C \left(\|\theta_0\|^2 + \iint_Q \hat{\sigma}^2 |f_0|^2 dxdt + \iint_Q \tilde{\rho}^2 |y|^2 dxdt \right.$$
$$\left. + \iint_Q \tilde{\rho}^2 |\theta|^2 dxdt \iint_{\omega \times (0,T)} \hat{\eta}^2 |v_0|^2 dxdt \right).$$

Proof. We have that

$$\zeta \leqslant C\tilde{\sigma},\ \zeta \leqslant C\hat{\sigma},\ \zeta \leqslant C\tilde{\eta},\ \zeta \leqslant C\hat{\eta},\ \zeta \leqslant C\tilde{\rho},\ |\zeta\zeta_t| \leqslant C\hat{\rho}\ .$$

Now, let us multiply $(2.12)_1$ by $\zeta^2 y$ and let us integrate in Ω, we obtain

$$\int_\Omega \zeta^2 (y_t - \nabla \cdot (\nu(t) Dy) + \nabla p) y dx = \int_\Omega \zeta^2 (f + v 1_\omega + \theta e_N) y dx.$$

Consequently,

$$\frac{1}{2}\frac{d}{dt} \int_\Omega \zeta^2 |y|^2 dx + \nu_0 \int_\Omega \zeta^2 |\nabla y|^2 dx \leqslant \int_\Omega \zeta^2 |f|^2 dx + \int_\omega \zeta^2 |v|^2 dx + \int_\Omega \zeta^2 |\theta|^2 dx$$
$$+ 3 \int_\Omega \zeta^2 |y|^2 dx$$
$$\leqslant C \left(\int_\Omega \tilde{\sigma}^2 |f|^2 dx + \int_\Omega \tilde{\eta}^2 |v|^2 dx + \int_\Omega \tilde{\rho}^2 |\theta|^2 dx \right.$$
$$\left. + \int_\Omega \tilde{\rho}^2 |y|^2 dx \right).$$

Integrating in time from 0 to t, with $t \leqslant T$, we get

$$\sup_{[0,T]} \int_\Omega \zeta^2 |y|^2 dx + \iint_Q \zeta^2 |\nabla y|^2 dxdt \leqslant C \left(\|y_0\|^2 + \iint_Q \tilde{\sigma}^2 |f|^2 dxdt + \iint_{\omega\times(0,T)} \tilde{\eta}^2 |v|^2 dxdt \right.$$
$$\left. + \iint_Q \tilde{\rho}^2 |\theta|^2 dxdt + \iint_Q \tilde{\rho}^2 |y|^2 dxdt \right).$$

Now, let us multiply $(2.12)_3$ by $\zeta^2\theta$ and let us integrate in Ω, we obtain

$$\int_\Omega \zeta^2 (\theta_t - \nabla\cdot(\nu(t)\nabla\theta))\theta dx = \int_\Omega \zeta^2 (f_0 + v_0 1_\omega)\theta dx.$$

Consequently,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \zeta^2|\theta|^2 dx + \nu_0 \int_\Omega \zeta^2 |\nabla\theta|^2 dx \leqslant \int_\Omega \zeta^2 |f_0|^2 dx + \int_\omega \zeta^2 |v_0|^2 dx + \int_\Omega \zeta^2 |\theta|^2 dx$$
$$\leqslant C\left(\int_\Omega \hat{\sigma}^2 |f_0|^2 dx + \int_\omega \hat{\eta}^2 |v_0|^2 dx + \int_\Omega \tilde{\rho}^2 |\theta|^2 dx\right).$$

Integrating in time from 0 to t, with $t \leqslant T$, we get

$$\sup_{[0,T]} \int_\Omega \zeta^2 |\theta|^2 dx + \iint_Q \zeta^2 |\nabla \theta|^2 dxdt \leqslant C \left(\|\theta_0\|^2 + \iint_Q \hat{\sigma}^2 |f_0|^2 dxdt + \iint_{\omega\times(0,T)} \hat{\eta}^2 |v_0|^2 dxdt \right.$$
$$\left. + \iint_Q \tilde{\rho}^2 |\theta|^2 dxdt \right).$$

□

Proposition 2.3.5. *Let the assumptions in Proposition* 2.3.3 *be satisfied and let* (y, p, θ, v) *satisfy* (2.12) *and* (2.16). *Then*

$$\mathit{sup}_{[0,T]} \int_\Omega \kappa^2 |\nabla y|^2 dx + \iint_Q \kappa^2 (|y_t|^2 + |\Delta y|^2) dxdt \leqslant C \left(\|y_0\|^2_{H^1_0} + \iint_Q \tilde{\sigma}^2 |f|^2 dxdt \right.$$
$$\left. + \iint_Q \tilde{\rho}^2 |y|^2 dxdt + \iint_Q \tilde{\rho}^2 |\theta|^2 dxdt \iint_{\omega\times(0,T)} \tilde{\eta}^2 |v|^2 dxdt \right)$$

and

$$\mathit{sup}_{[0,T]} \int_\Omega \kappa^2 |\nabla \theta|^2 dx + \iint_Q \kappa^2 (|\theta_t|^2 + |\Delta \theta|^2) dxdt \leqslant C \left(\|\theta_0\|^2_{H^1_0} + \iint_Q \hat{\sigma}^2 |f_0|^2 dxdt \right.$$
$$\left. + \iint_Q \tilde{\rho}^2 |y|^2 dxdt + \iint_Q \tilde{\rho}^2 |\theta|^2 dxdt \iint_{\omega\times(0,T)} \hat{\eta}^2 |v_0|^2 dxdt \right).$$

Proof. We have that

$$\kappa \leqslant C\zeta, \ |\kappa\kappa_t| \leqslant C\zeta^2.$$

Now, let us multiply $(2.12)_1$ by $\kappa^2 y_t$ and let us integrate in Ω, we obtain

$$\int_\Omega \kappa^2 (y_t - \nabla\cdot(\nu(t)Dy) + \nabla p) y_t dx = \int_\Omega \kappa^2 (f + v1_\omega + \theta e_N) y_t dx.$$

Consequently,

$$\int_\Omega \kappa^2|y_t|^2dx+\frac{1}{2}\frac{d}{dt}\int_\Omega \kappa^2\nu(t)|\nabla y|^2dx \leqslant \epsilon\int_\Omega \kappa^2|y_t|^2dx + C\int_\Omega \zeta^2|\nabla y|^2dx$$
$$+C_\epsilon\left(\int_\Omega \kappa^2|f|^2dx+\int_\omega \kappa^2|v|^2dx+\int_\Omega \kappa^2|\theta|^2dx\right)$$

then

$$\int_\Omega \kappa^2|y_t|^2dx+\frac{d}{dt}\int_\Omega \kappa^2\nu(t)|\nabla y|^2dx \leqslant C\left(\int_\Omega \tilde{\sigma}^2|f|^2dx+\int_\Omega \tilde{\rho}^2|\theta|^2dx\right.$$
$$\left.+\int_\omega \tilde{\eta}^2|v|^2dx+\int_\Omega \zeta^2|\nabla y|^2dx\right).$$

Integrating in time from 0 to t with $t\leqslant T$ and using the Proposition 2.3.4, we get

$$\sup_{[0,T]}\int_\Omega \kappa^2|\nabla y|^2dx+\iint_Q \kappa^2|y_t|^2dxdt \leqslant\ C\ \left(\|y_0\|^2_{H^1_0}+\iint_Q \tilde{\sigma}^2|f|^2dxdt\right.$$
$$\left.+\iint_Q \tilde{\rho}^2|y|^2dxdt+\iint_Q \tilde{\rho}^2|\theta|^2dxdt\iint_{\omega\times(0,T)} \tilde{\eta}^2|v|^2dxdt\right).$$

Now, let us multiply $(2.12)_1$ by $\kappa^2 Ay$ and let us integrate in Ω, we obtain

$$\int_\Omega \kappa^2(y_t-\nabla\cdot(\nu(t)Dy)+\nabla p)Aydx=\int_\Omega \kappa^2(f+v1_\omega+\theta e_N)Aydx.$$

Consequently,

$$\int_\Omega \kappa^2\nu(t)|\Delta y|^2dx \leqslant \epsilon\int_\Omega \kappa^2\nu(t)|\Delta y|^2dx-\int_\Omega \kappa^2y_t.Aydx$$
$$+C_\epsilon\left(\int_\Omega \kappa^2|f|^2dx+\int_\omega \kappa^2|v|^2dx+\int_\Omega \kappa^2|\theta|^2dx\right)$$
$$\leqslant \epsilon\int_\Omega \kappa^2\nu(t)|\Delta y|^2dx+C_\epsilon\left(\int_\Omega \tilde{\sigma}^2|f|^2dx+\int_\Omega \tilde{\rho}^2|\theta|^2dx\right.$$
$$\left.+\int_\omega \tilde{\eta}^2|v|^2dx+\int_\Omega \kappa^2|y_t|^2dx\right)$$

thus

$$\int_\Omega \kappa^2|\Delta y|^2dx \leqslant C\left(\int_\Omega \tilde{\sigma}^2|f|^2dx+\int_\Omega \tilde{\rho}^2|\theta|^2dx+\int_\omega \tilde{\eta}^2|v|^2dx+\int_\Omega \kappa^2|y_t|^2dx\right).$$

Integrating in time from 0 to t, with $t\leqslant T$, we get

$$\iint_Q \kappa^2|\Delta y|^2dx \leqslant C\left(\iint_Q \tilde{\sigma}^2|f|^2dx+\iint_Q \tilde{\rho}^2|\theta|^2dx+\iint_{Q_\omega} \tilde{\eta}^2|v|^2dx+\iint_Q \kappa^2|y_t|^2dx\right).$$

Now, let us multiply $(2.12)_3$ by $\kappa^2\theta_t$ and let us integrate in Ω, we obtain

$$\int_\Omega \kappa^2(\theta_t-\nabla\cdot(\nu(t)\nabla\theta))\theta_tdx=\int_\Omega \kappa^2(f_0+v_01_\omega)\theta_tdx.$$

Consequently,

$$\int_\Omega \kappa^2|\theta_t|^2dx + \frac{1}{2}\frac{d}{dt}\int_\Omega \kappa^2\nu(t)|\nabla\theta|^2dx \leqslant \epsilon\int_\Omega \kappa^2|\theta_t|^2dx + C_\epsilon\left(\int_\Omega \kappa^2|f_0|^2dx + \int_\omega \kappa^2|v_0|^2dx\right) + C\int_\Omega \zeta^2|\nabla\theta|^2dx$$

then

$$\int_\Omega \kappa^2|\theta_t|^2dx + \frac{d}{dt}\int_\Omega \kappa^2\nu(t)|\nabla\theta|^2dx \leqslant C\left(\int_\Omega \hat{\sigma}^2|f_0|^2dx + \int_\omega \hat{\eta}^2|v_0|^2dx + \int_\Omega \zeta^2|\nabla\theta|^2dx\right).$$

Integrating in time from 0 to t, with $t \leqslant T$ and using the Proposition 2.3.4, we get

$$\sup_{[0,T]}\int_\Omega \kappa^2|\nabla\theta|^2dx + \iint_Q \kappa^2|\theta_t|^2dxdt \leqslant C\left(\|\theta_0\|^2_{H^1_0} + \iint_Q \hat{\sigma}^2|f_0|^2dxdt + \iint_Q \tilde{\rho}^2|\theta|^2dxdt \iint_{\omega\times(0,T)} \hat{\eta}^2|v_0|^2dxdt\right).$$

Now, let us multiply $(2.12)_3$ by $\kappa^2(-\Delta\theta)$ and let us integrate in Ω, we obtain

$$\int_\Omega \kappa^2(-\nabla\cdot(\nu(t)\nabla\theta))(-\Delta\theta)dx = \int_\Omega \kappa^2(f_0 + v_0 1_\omega - \theta_t)(-\Delta\theta)dx.$$

Consequently,

$$\int_\Omega \kappa^2\nu(t)|\Delta\theta|^2dx \leqslant \epsilon\int_\Omega \kappa^2|\Delta\theta|^2dx + C_\epsilon\left(\int_\Omega \kappa^2|f_0|^2dx + \int_\omega \kappa^2|v_0|^2dx + \int_\Omega \kappa^2|\theta_t|^2dx\right)$$

then

$$\int_\Omega \kappa^2|\Delta\theta|^2dx \leqslant C\left(\int_\Omega \hat{\sigma}^2|f_0|^2dx + \int_\omega \hat{\eta}^2|v_0|^2dx + \int_\Omega \kappa^2|\theta_t|^2dx\right).$$

Integrating in time from 0 to t, with $t \leqslant T$, we get

$$\iint_Q \kappa^2|\Delta\theta|^2dxdt \leqslant C\left(\iint_Q \hat{\sigma}^2|f_0|^2dxdt + \iint_{Q_\omega} \hat{\eta}^2|v_0|^2dxdt + \iint_Q \kappa^2|\theta_t|^2dxdt + \int_\Omega \kappa^2|\nabla\theta|^2dxdt\right).$$

Then, we deduce

$$\iint_Q \kappa^2|\Delta\theta|^2dx \leqslant C\left(\|\theta_0\|^2_{H^1_0} + \iint_Q \hat{\sigma}^2|f_0|^2dxdt + \iint_Q \tilde{\rho}^2|y|^2dxdt + \iint_Q \tilde{\rho}^2|\theta|^2dxdt \iint_{\omega\times(0,T)} \hat{\eta}^2|v_0|^2dxdt\right).$$

$\square$

2.4 The proof of Theorem 2.1.1

In this section, we will prove the local null controllability of the system (2.1). Our aim is apply Liusternik's Inverse Function Theorem in infinite dimensional spaces.

Thus, let us introduce the space

$$\begin{aligned} Y_N = \big\{ &(y,p,v,\theta,v_0) : v_N \equiv 0,\ v_j \equiv 0, \text{ for one } j < N;\ \tilde{\rho} y, \tilde{\eta} v 1_\omega \in L^2(Q)^N; \\ &\tilde{\rho}\theta, \hat{\eta} v_0 1_\omega \in L^2(Q);\ y \in L^2(0,T;D(A)), \theta \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \\ &p \in L^2(0,T;H^1(\Omega));\ \tilde{\sigma}(y_t - \nabla.(\nu(t)Dy) + \nabla p - \theta e_N - v1_\omega) \in L^2(Q)^N, \\ &\hat{\sigma}(\theta_t - \nabla.(\nu(t)\nabla\theta) - v_0 1_\omega) \in L^2(Q) \big\}. \end{aligned}$$

It is clear that $Y_N \neq \emptyset$ (using the linearized problem) is a Banach space for the norm $\|\cdot\|_{Y_N}$, where

$$\begin{aligned} \|(y,p,v,\theta,v_0)\|_{Y_N}^2 = &\|\tilde{\rho} y\|_{L^2(Q)^N}^2 + \|\tilde{\eta} v\|_{L^2(\omega\times(0,T))^N}^2 + \|\tilde{\rho}\theta\|_{L^2(Q)}^2 + \|\hat{\eta} v_0\|_{L^2(\omega\times(0,T))}^2 \\ &+ \|y\|_{L^2(0,T;D(A))}^2 + \|\theta\|_{L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)}^2 + \|p\|_{L^2(0,T;H^1(\Omega))}^2 \\ &+ \|\tilde{\sigma}(y_t - \nabla.(\nu(t)Dy) + \nabla p - \theta e_N - v1_\omega)\|_{L^2(Q)^N}^2 \\ &+ \|\hat{\sigma}(\theta_t - \nabla.(\nu(t)\nabla\theta) - v_0 1_\omega)\|_{L^2(Q)}^2. \end{aligned}$$

Notice that, if $(y,p,v,\theta,v_0) \in Y_N$, then $y_t \in L^2(Q)^N$, $\theta_t \in L^2(Q)$, whence $y : [0,T] \mapsto V$ and $\theta : [0,T] \mapsto H_0^1(\Omega)$ are continuous and, in particular, we have $y(0) \in V$, $\theta(0) \in H_0^1(\Omega)$, and also

$$\|y(0)\|_V^2 \leqslant C\|(y,p,v,\theta,v_0)\|_{Y_N}^2 \quad \text{and} \quad \|\theta(0)\|_{H_0^1(\Omega)}^2 \leqslant C\|(y,p,v,\theta,v_0)\|_{Y_N}^2.$$

Furthermore, in view of Propositions 2.3.4 and 2.3.5, one also has

$$\|\zeta y\|_{L^2(0,T;V)\cap L^\infty(0,T;H)}^2 + \|\zeta\theta\|_{L^2(0,T;H_0^1)\cap L^\infty(0,T;L^2)}^2 \leqslant C\|(y,p,v,\theta,v_0)\|_{Y_N}^2, \tag{2.17}$$

and

$$\|\kappa y\|_{L^2(0,T;D(A))\cap L^\infty(0,T;V)}^2 + \|\kappa\theta\|_{L^2(0,T;H_0^1\cap H^2)\cap L^\infty(0,T;H_0^1)}^2 \leqslant C\|(y,p,v,\theta,v_0)\|_{Y_N}^2. \tag{2.18}$$

Let us introduce the Banach space

$$Z_N = L^2(\tilde{\sigma}^2;Q)^N \times V \times L^2(\hat{\sigma}^2;Q) \times H_0^1(\Omega)$$

and the mapping

$$\mathcal{A} : Y_N \longrightarrow Z_N$$
$$\mathcal{A}(y,p,v,\theta,v_0) = (\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\mathcal{A}_4)(y,p,v,\theta,v_0)$$

where

$$\begin{aligned}
&\mathcal{A}_1(y,p,v,\theta,v_0) = y_t - \nabla\cdot(\nu_0+\nu_1(\|Dy\|^2,x,t)Dy) + (y\cdot\nabla)y + \nabla p - v1_\omega - \theta e_N,\\
&\mathcal{A}_2(y,p,v,\theta,v_0) = y(0),\\
&\mathcal{A}_3(y,p,v,\theta,v_0) = \theta_t - \nabla\cdot(\nu_0+\nu_1(\|Dy\|^2,x,t)\nabla\theta) + y\cdot\nabla\theta - v_0 1_\omega,\\
&\mathcal{A}_4(y,p,v,\theta,v_0) = \theta(0).
\end{aligned}$$

Lemma 2.4.1. *$\mathcal{A}$ is well defined and continuous.*

Proof. First, note that, in view of (2.8) and (2.15), we have

$$\tilde\sigma^2 \leqslant C\zeta\kappa^3 \leqslant C\kappa^6, \quad \hat\sigma^2 \leqslant C\zeta\kappa^3.$$

Let us see that, if $(y,p,v,\theta,v_0)\in Y_N$ then $\mathcal{A}_1(y,p,v,\theta,v_0)\in L^2(\tilde\sigma^2;Q)^N$. Indeed, one has

$$\begin{aligned}
\iint_Q &\tilde\sigma^2|y_t - \nabla\cdot(\nu_0+\nu_1(\|Dy\|^2,x,t)Dy) + (y\cdot\nabla)y + \nabla p - v1_\omega - \theta e_N|^2 dxdt\\
&\leqslant 3\iint_Q \tilde\sigma^2|y_t - \nabla\cdot(\nu(t)Dy) + \nabla p - v1_\omega - \theta e_N|^2 dxdt\\
&\quad + 3\iint_Q \tilde\sigma^2|\nabla\cdot((\nu_1(\|Dy\|^2,x,t)-\nu_1(0,x,t))Dy)|^2 dxdt\\
&\quad + 3\iint_Q \tilde\sigma^2|(y\cdot\nabla)y|^2 dxdt\\
&\quad := 3M_1 + 3M_2 + 3M_3.
\end{aligned}$$

From the definition of Y_N, we have

$$M_1 \leqslant \|(y,p,v,\theta,v_0)\|^2_{Y_N}.$$

On the other hand, from Proposition 2.3.5 we deduce that

$$
\begin{aligned}
M_2 \leqslant C & \left(\iint_Q \tilde{\sigma}^2 |\nabla(\nu_1(\|Dy\|^2, x, t) - \nu_1(0, x, t))|^2 |\nabla y|^2 dxdt \right. \\
& + \left. \iint_Q \tilde{\sigma}^2 |\nu_1(\|Dy\|^2, x, t) - \nu_1(0, x, t)|^2 |\Delta y|^2 dxdt \right) \\
& \leqslant C \left(\iint_Q \kappa^6 \|\nabla y\|^4 |\nabla y|^2 dxdt + \iint_Q \kappa^6 \|\nabla y\|^4 |\Delta y|^2 dxdt \right) \\
& \leqslant C \left(\left\{ \underset{[0,T]}{sup}\ \kappa^2 \|\nabla y\|^2 \right\}^3 + \left\{ \underset{[0,T]}{sup}\ \kappa^2 \|\nabla y\|^2 \right\}^2 \iint_Q \kappa^2 |\Delta y|^2 dxdt \right) \\
& \leqslant C \|(y, p, v, \theta, v_0)\|_{Y_N}^6 .
\end{aligned}
$$

Finally, taking into account that $\|\nabla w\|_{L^3} \leqslant C\|\nabla w\|^{1/2}\|\Delta w\|^{1/2}$ and

$$
\|(w \cdot \nabla)w\|^2 \leqslant C\|w\|_{L^6}^2 \|\nabla w\|_{L^3}^2 \leqslant C\|\nabla w\|^3 \|\Delta w\|
$$

for all $w \in D(A)$, we have

$$
\begin{aligned}
M_3 \leqslant C & \int_0^T \zeta \kappa^3 \|(y \cdot \nabla)y\|^2 dt \\
& \leqslant C \int_0^T \zeta \kappa^3 \|\nabla y\|^3 \|\Delta y\| dt \\
& \leqslant C \left(\underset{[0,T]}{sup}\ \kappa \|\nabla y\| \right)^2 \int_0^T \zeta \|\nabla y\| \kappa \|\Delta y\| dt \\
& \leqslant C \|\kappa y\|_{L^\infty(0,T;V)}^2 \left(\int_0^T \zeta^2 \|\nabla y\|^2 dt \right)^{1/2} \left(\int_0^T \kappa^2 \|\Delta y\|^2 dt \right)^{1/2} \\
& \leqslant C \|(y, p, v, \theta, v_0)\|_{Y_N}^4 .
\end{aligned}
$$

Now, let see that, if $(y, p, v, \theta, v_0) \in Y_N$, then $\mathcal{A}_3(y, p, v, \theta, v_0) \in L^2(\hat{\sigma}^2; Q)$. Indeed, one has

$$
\begin{aligned}
\iint_Q \hat{\sigma}^2 |\theta_t - & \nabla \cdot (\nu_0 + \nu_1(\|Dy\|^2, x, t)\nabla\theta) + y \cdot \nabla\theta - v_0 1_\omega|^2 dxdt \\
& \leqslant 3 \iint_Q \hat{\sigma}^2 |\theta_t - \nabla \cdot (\nu(t)\nabla\theta) - v_0 1_\omega|^2 dxdt \\
& + 3 \iint_Q \tilde{\sigma}^2 |\nabla \cdot (\nu_1(\|Dy\|^2, x, t) - \nu_1(0, x, t))\nabla\theta)|^2 dxdt \\
& + 3 \iint_Q \hat{\sigma}^2 |y \cdot \nabla\theta|^2 dxdt \\
& := 3N_1 + 3N_2 + 3N_3 .
\end{aligned}
$$

From the definition of Y_N, we have

$$N_1 \leqslant \|(y,p,v,\theta,v_0)\|^2_{Y_N}.$$

On the other hand, from Proposition 2.3.5 we deduce that

$$\begin{aligned}
N_2 \leqslant & C\left(\iint_Q \tilde{\sigma}^2|\nabla(\nu_1(\|Dy\|^2,x,t)-\nu_1(0,x,t))|^2|\nabla\theta|^2dxdt\right.\\
&\left.+\iint_Q \tilde{\sigma}^2|\nu_1(\|Dy\|^2,x,t)-\nu_1(0,x,t)|^2|\Delta\theta|^2dxdt\right)\\
&\leqslant C\left(\iint_Q \kappa^6\|\nabla y\|^4|\nabla\theta|^2dxdt+\iint_Q \kappa^6\|\nabla y\|^4|\Delta\theta|^2dxdt\right)\\
&\leqslant C\left(\left\{\underset{[0,T]}{sup}\ \kappa^2\|\nabla y\|^2\right\}^2\left\{\underset{[0,T]}{sup}\ \kappa^2\|\nabla\theta\|^2\right\}+\left\{\underset{[0,T]}{sup}\ \kappa^2\|\nabla y\|^2\right\}^2\iint_Q \kappa^2|\Delta\theta|^2dxdt\right)\\
&\leqslant C\|(y,p,v,\theta,v_0)\|^6_{Y_N}.
\end{aligned}$$

Finally, taking into account that $\|\nabla w\|_{L^3}\leqslant C\|\nabla w\|^{1/2}\|\Delta w\|^{1/2}$ and

$$\|z\cdot\nabla w\|^2\leqslant C\|z\|^2_{L^6}\|\nabla w\|^2_{L^3}\leqslant C\|\nabla z\|^2\|\nabla w\|\|\Delta w\|$$

for all $z\in(H^2(\Omega)\cap H^1_0(\Omega))^3$ and $w\in H^2(\Omega)\cap H^1_0(\Omega)$, we have

$$\begin{aligned}
N_3 &\leqslant C\int_0^T \zeta\kappa^3\|y\cdot\nabla\theta\|^2dt\\
&\leqslant C\int_0^T \zeta\kappa^3\|\nabla y\|^2\|\nabla\theta\|\|\Delta\theta\|dt\\
&\leqslant C\left(\underset{[0,T]}{sup}\ \kappa\|\nabla y\|\right)^2\int_0^T \zeta\|\nabla\theta\|\kappa\|\Delta\theta\|dt\\
&\leqslant C\|\kappa y\|^2_{L^\infty(0,T;V)}\left(\int_0^T \zeta^2\|\nabla\theta\|^2dt\right)^{1/2}\left(\int_0^T \kappa^2\|\Delta\theta\|^2dt\right)^{1/2}\\
&\leqslant C\|(y,p,v,\theta,v_0)\|^4_{Y_N}.
\end{aligned}$$

Consequently $\mathcal{A}$ takes values in Z_N.

Let us see that $\mathcal{A}$ is continuous, thus, let us assume that $(y_n,p_n,v_n,\theta_n,v_{0n})\overset{E_N}{\to}(y,p,v,\theta,v_0)$ and let us see that $\mathcal{A}(y_n,p_n,v_n,\theta_n,v_{0n})\overset{Z_N}{\to}\mathcal{A}(y,p,v,\theta,v_0)$.

Obviously, $y_n(\cdot,0) \xrightarrow{V} y(\cdot,0)$ and $\theta_n(\cdot,0) \xrightarrow{H_0^1(\Omega)} \theta(\cdot,0)$. Moreover

$$
\begin{aligned}
&\iint_Q \tilde{\sigma}^2|\mathcal{A}_1((y_n,p_n,v_n,\theta_n,v_{0n}) - \mathcal{A}_1(y,p,v,\theta,v_0)|^2dxdt\\
\leqslant& 3\iint_Q \tilde{\sigma}^2|(y_n-y)_t - \nabla\cdot(\nu(t)D(y_n-y)) + \nabla(p_n-p) - (v_n-v)1_\omega - (\theta_n-\theta)e_N|^2dxdt\\
&+3\iint_Q \tilde{\sigma}^2|\nabla\cdot((\nu_1(\|Dy_n\|^2,x,t)-\nu_1(0,x,t))Dy_n) - \nabla\cdot((\nu_1(\|Dy\|^2,x,t)-\nu_1(0,x,t))Dy)|^2dxdt\\
&+3\iint_Q \tilde{\sigma}^2|(y_n\cdot\nabla)y_n - (y\cdot\nabla)y|^2dxdt\\
:=& 3Z_{1,n} + 3Z_{2,n} + 3Z_{3,n}.
\end{aligned}
$$

By definition of Y_N, we have

$$
Z_{1,n} \leqslant \|(y_n,p_n,v_n,\theta_n,v_{0n}) - (y,p,v,\theta,v_0)\|_{Y_N}^2.
$$

Using Proposition 2.3.5 and properties of ν_1, we have

$$
\begin{aligned}
Z_{2,n} \leqslant& 2\iint_Q \tilde{\sigma}^2|\nabla\cdot((\nu_1(\|Dy_n\|^2,x,t)-\nu_1(\|Dy\|^2,x,t))Dy_n)|^2dxdt\\
&+2\iint_Q \tilde{\sigma}^2|\nabla\cdot((\nu_1(\|Dy\|^2,x,t)-\nu_1(0,x,t))D(y_n-y)|^2dxdt\\
\leqslant& C\iint_Q \tilde{\sigma}^2|\nabla(\nu_1(\|\nabla y_n\|^2,x,t)-\nu_1(\|\nabla y\|^2,x,t))|^2|\nabla y_n|^2dxdt\\
&+C\iint_Q \tilde{\sigma}^2|\nu_1(\|Dy_n\|^2,x,t)-\nu_1(\|Dy\|^2,x,t)|^2|\Delta y_n|^2dxdt\\
&+C\iint_Q \tilde{\sigma}^2|\nabla(\nu_1(\|Dy\|^2,x,t)-\nu_1(0,x,t))|^2|\nabla(y_n-y)|^2dxdt\\
&+C\iint_Q \tilde{\sigma}^2|\nu_1(\|Dy\|^2,x,t)-\nu_1(0,x,t)|^2|\Delta(y_n-y)|^2dxdt\\
\leqslant& C\left(\iint_Q \kappa^6|\|Dy_n\|^2-\|Dy\|^2|^2(|\nabla y_n|^2+|\Delta y_n|^2)dxdt\right.\\
&\left.+\iint_Q \kappa^6\|\nabla y\|^4(|\nabla(y_n-y)|^2+|\Delta(y_n-y)|^2)dxdt\right)\\
\leqslant& C\left(\iint_Q \kappa^6(\|\nabla y_n\|^2+\|\nabla y\|^2)\|\nabla(y_n-y)\|^2(|\nabla y_n|^2+|\Delta y_n|^2)dxdt\right.\\
&\left.+\iint_Q \kappa^6\|\nabla y\|^4(|\nabla(y_n-y)|^2+|\Delta(y_n-y)|^2)dxdt\right)\\
\leqslant& C\|(y_n,p_n,v_n,\theta_n,v_{0n}) - (y,p,v,\theta,v_0)\|_{Y_N}^2
\end{aligned}
$$

where

$$
C := C\left(\|(y_n,p_n,v_n,\theta_n,v_{0n})\|_{Y_N}^4 + \|(y,p,v,\theta,v_0)\|_{Y_N}^4\right).
$$

Similarly the estimates for M_3

$$Z_{3,n} \leqslant C\|(y_n,p_n,v_n,\theta_n,v_{0n})-(y,p,v,\theta,v_0)\|^2_{Y_N}$$

where

$$C := C\left(\|(y_n,p_n,v_n,\theta_n,v_{0n})\|^2_{Y_N}+\|(y,p,v,\theta,v_0)\|^2_{Y_N}\right).$$

This shows that $\mathcal{A}_1$ is continuous.

Now, we get

$$\begin{aligned}
&\iint_Q \tilde{\sigma}^2|\mathcal{A}_3((y_n,p_n,v_n,\theta_n,v_{0n})-\mathcal{A}_3(y,p,v,\theta,v_0)|^2dxdt\\
&\quad\leqslant 3\iint_Q \hat{\sigma}^2|(\theta_n-\theta)_t-\nabla\cdot(\nu(t)\nabla(\theta_n-\theta))-(v_{0n}-v_0)1_\omega|^2dxdt\\
&\qquad+3\iint_Q \tilde{\sigma}^2|\nabla\cdot((\nu_1(\|Dy_n\|^2,x,t)-\nu_1(0,x,t))\nabla\theta_n)-\nabla\cdot((\nu_1(\|Dy\|^2,x,t)-\nu_1(0,x,t))\nabla\theta)|^2dxd\\
&\qquad+3\iint_Q \hat{\sigma}^2|y_n\cdot\nabla\theta_n-y\cdot\nabla\theta|^2dxdt\\
&\quad:=3\tilde{Z}_{1,n}+3\tilde{Z}_{2,n}+3\tilde{Z}_{3,n}.
\end{aligned}$$

By definition of Y_N, we have

$$\tilde{Z}_{1,n}\leqslant\|(y_n,p_n,v_n,\theta_n,v_{0n})-(y,p,v,\theta,v_0)\|^2_{Y_N}.$$

On the other hand, similarly to $Z_{2,n}$ we deduce

$$\begin{aligned}
\tilde{Z}_{2,n}&\leqslant C\left(\iint_Q\kappa^6(\|\nabla y_n\|^2+\|\nabla y\|^2)\|\nabla(y_n-y)\|^2(|\nabla\theta_n|^2+|\Delta\theta_n|^2)dxdt\right.\\
&\qquad\left.+\iint_Q\kappa^6\|\nabla y\|^4(|\nabla(\theta_n-\theta)|^2+|\Delta(\theta_n-\theta)|^2)dxdt\right)\\
&\leqslant C\|(y_n,p_n,v_n,\theta_n,v_{0n})-(y,p,v,\theta,v_0)\|^2_{Y_N}
\end{aligned}$$

where

$$C := C\left(\|(y_n,p_n,v_n,\theta_n,v_{0n})\|^4_{Y_N}+\|(y,p,v,\theta,v_0)\|^4_{Y_N}\right)$$

and finally

$$\begin{aligned}
\tilde{Z}_{3,n}&\leqslant 2\iint_Q\hat{\sigma}^2|(y_n-y)\cdot\nabla\theta_n|^2dxdt+2\iint_Q\hat{\sigma}^2|y\cdot(\nabla\theta_n-\nabla\theta)|^2dxdt\\
&\leqslant C\int_0^T\zeta\kappa^3\|(y_n-y)\cdot\nabla\theta_n\|^2dt+C\int_0^T\zeta\kappa^3\|y\cdot\nabla(\theta_n-\theta)\|^2dt\\
&\leqslant C\int_0^T\kappa^2\|\nabla(y_n-y)\|^2\zeta\|\nabla\theta_n\|\kappa\|\Delta\theta\|dt+C\int_0^T\kappa^2\|\nabla y\|^2\zeta\|\nabla(\theta_n-\theta)\|\kappa\|\Delta(\theta_n-\theta)\|dt\\
&\leqslant C\|(y_n,p_n,v_n,\theta_n,v_{0n})-(y,p,v,\theta,v_0)\|^2_{Y_N}
\end{aligned}$$

where

$$C := C\left(\|(y_n,p_n,v_n,\theta_n,v_{0n})\|_{Y_N}^2 + \|(y,p,v,\theta,v_0)\|_{Y_N}^2\right).$$

This shows that $\mathcal{A}_3$ is continuous and ends the proof. $\square$

Lemma 2.4.2. *$\mathcal{A}$ is continuously differentiable.*

Proof. Let us first prove that $\mathcal{A}$ is $G-$differentiable at any $(y,p,v,\theta,v_0)\in Y_N$ and let us compute the $G-$derivative $\mathcal{A}'(y,p,v,\theta,v_0)$.
Thus, let us fix $(y,p,v,\theta,v_0)\in Y_N$ and let us take $(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0)\in Y_N$ and $\lambda \geqslant 0$. For simplicity, we will use the notation

$$\nu_{1,\lambda} := \nu_1(\|Dy+\lambda D\tilde{y}\|^2,x,t), \quad \nu_{1,\lambda}' := \partial_1\nu_1(\|Dy+\lambda D\tilde{y}\|^2,x,t),$$

$$\tilde{\nu}_{1,n} := \nu_1(\|Dy_n\|^2,x,t), \quad \tilde{\nu}_{1,n}' := \partial_1\nu_1(\|Dy_n\|^2,x,t).$$

We have,

$$\begin{aligned}
&\frac{1}{\lambda}\left[\mathcal{A}_1((y,p,v,\theta,v_0)+\lambda(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0)) - \mathcal{A}_1(y,p,v,\theta,v_0)\right]\\
=&\tilde{y}_t - \nabla\cdot((\nu_0+\nu_{1,\lambda})D\tilde{y}) - \frac{1}{\lambda}(\nabla\cdot((\nu_{1,\lambda}-\nu_{1,0})Dy) + \nabla\tilde{p} + (\tilde{y}\cdot\nabla)y\\
&+ (y\cdot\nabla)\tilde{y} + \lambda(\tilde{y}\cdot\nabla)\tilde{y} - \tilde{v}1_\omega - \tilde{\theta}e_N,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\lambda}\left[\mathcal{A}_3((y,p,v,\theta,v_0)+\lambda(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0)) - \mathcal{A}_3(y,p,v,\theta,v_0)\right]\\
=&\tilde{\theta}_t - \nabla\cdot((\nu_0+\nu_{1,\lambda})\nabla\tilde{\theta}) - \frac{1}{\lambda}(\nabla\cdot(\nu_{1,\lambda}-\nu_{1,0})\nabla\theta) - \tilde{v}_0 1_\omega + \tilde{y}\cdot\nabla\theta + y\cdot\nabla\tilde{\theta} + \lambda\tilde{y}\cdot\nabla\tilde{\theta}.
\end{aligned}$$

Let us introduce the linear mapping

$$\begin{aligned}
D\mathcal{A}(y,p,v,\theta,v_0) : Y_N &\longrightarrow Z_N\\
(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0) &\mapsto D\mathcal{A}(y,p,v,\theta,v_0)(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0)
\end{aligned}$$

where

$$\begin{aligned}
D\mathcal{A}_1(y,p,v,\theta,v_0)(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0) =&\tilde{y}_t - \nabla\cdot((\nu_0+\nu_{1,0})D\tilde{y}) - 2(Dy,D\tilde{y})\nabla\cdot(\nu_{1,0}'Dy) + \nabla\tilde{p},\\
&+ (\tilde{y}\cdot\nabla)y + (y\cdot\nabla)\tilde{y} - \tilde{v}1_\omega - \tilde{\theta}e_N,\\
D\mathcal{A}_2(y,p,v,\theta,v_0)(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0) =&\tilde{y}(0),\\
D\mathcal{A}_3(y,p,v,\theta,v_0)(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0) =&\tilde{\theta}_t - \nabla\cdot((\nu_0+\nu_{1,0})\nabla\tilde{\theta}) - 2(Dy,D\tilde{y})\nabla\cdot(\nu_{1,0}'\nabla\theta) - \tilde{v}_0 1_\omega\\
&+ \tilde{y}\cdot\nabla\theta + y\cdot\nabla\tilde{\theta}\\
D\mathcal{A}_4(y,p,v,\theta,v_0)(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0) =&\tilde{\theta}(0).
\end{aligned}$$

From the definition of the spaces Y_N and Z_N, it becomes clear that $D\mathcal{A}(y,p,v,\theta,v_0) \in \mathcal{L}(Y_N, Z_N)$. Furthermore,

$$\frac{1}{\lambda}\Big[\mathcal{A}_1((y,p,v,\theta,v_0)+\lambda(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0)) - \mathcal{A}_1(y,p,v,\theta,v_0)\Big] \overset{L^2(\tilde{\sigma}^2;Q)}{\longrightarrow} D\mathcal{A}_1(y,p,v,\theta,v_0)$$

and

$$\frac{1}{\lambda}\Big[\mathcal{A}_3((y,p,v,\theta,v_0)+\lambda(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0)) - \mathcal{A}_3(y,p,v,\theta,v_0)\Big] \overset{L^2(\hat{\sigma}^2;Q)}{\longrightarrow} D\mathcal{A}_3(y,p,v,\theta,v_0)$$

as $\lambda \to 0$.

Indeed, we have

$$\begin{aligned}
&\|\frac{1}{\lambda}\Big[\mathcal{A}_1((y,p,v,\theta,v_0)+\lambda(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0)) - \mathcal{A}_1(y,p,v,\theta,v_0)\Big] - D\mathcal{A}_1(y,p,v,\theta,v_0)\|^2_{L^2(\tilde{\sigma}^2;Q)}\\
&= \|\nabla\cdot((\nu_{1,0}-\nu_{1,\lambda})D\tilde{y}) + \nabla\cdot(\Big[2\nu'_{1,0}(Dy,D\tilde{y}) - \frac{1}{\lambda}(\nu_{1,\lambda}-\nu_{1,0})\Big]Dy) + \lambda(\tilde{y}\cdot\nabla)\tilde{y}\|^2_{L^2(\tilde{\sigma}^2;Q)}\\
&\leqslant 3\|\nabla\cdot((\nu_{1,\lambda}-\nu_{1,0})D\tilde{y})\|^2 + 3\|\nabla\cdot(\Big[2\nu'_{1,0}(Dy,D\tilde{y}) - \frac{1}{\lambda}(\nu_{1,\lambda}-\nu_{1,0})\Big]Dy)\|^2 + 3\|\lambda(\tilde{y}\cdot\nabla)\tilde{y}\|^2\\
&:= 3B_1 + 3B_2 + 3B_3.
\end{aligned}$$

Estimative for B_1

$$\begin{aligned}
B_1 &:= \iint_Q \tilde{\sigma}^2|\nabla\cdot((\nu_{1,\lambda}-\nu_{1,0})D\tilde{y})|^2 dxdt\\
&\leqslant 2\iint_Q \tilde{\sigma}^2|\nabla(\nu_{1,\lambda}-\nu_{1,0})|^2|\nabla\tilde{y}|^2 dxdt + 2\iint_Q \tilde{\sigma}^2|\nu_{1,\lambda}-\nu_{1,0}|^2|\Delta\tilde{y}|^2 dxdt\\
&\leqslant C\left(\iint_Q \tilde{\sigma}^2|\|Dy+\lambda D\tilde{y}\|^2 - \|Dy\|^2|^2(|\nabla\tilde{y}|^2+|\Delta\tilde{y}|^2)dxdt\right)\\
&= C\left(\iint_Q \tilde{\sigma}^2|2\lambda(Dy,D\tilde{y}) + \lambda^2\|D\tilde{y}\|^2|^2(|\nabla\tilde{y}|^2+|\Delta\tilde{y}|^2)dxdt\right)\\
&\leqslant C\left(\iint_Q \kappa^6(4\lambda^2\|\nabla y\|^2\|\nabla\tilde{y}\|^2 + \lambda^4\|\nabla\tilde{y}\|^4)(|\nabla\tilde{y}|^2+|\Delta\tilde{y}|^2)dxdt\right)\\
&\leqslant \lambda^2 C\left(\iint_Q \kappa^6(2\|\nabla y\|^4 + (2+\lambda^2)\|\nabla\tilde{y}\|^4)(|\nabla\tilde{y}|^2+|\Delta\tilde{y}|^2)dxdt\right)\\
&\leqslant \lambda^2 C(\|(y,p,v,\theta,v_0)\|^4_{Y_N} + \|(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0)\|^4_{Y_N})\|(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0)\|^2_{Y_N}
\end{aligned}$$

where $B_1 \to 0$, as $\lambda \to 0$.

Estimative for B_2

$$\begin{aligned}
B_2 :=& \iint_Q \tilde{\sigma}^2 |\nabla \cdot (\left[2\nu'_{1,0}(\nabla y, \nabla \tilde{y}) - \frac{1}{\lambda}(\nu_{1,\lambda} - \nu_{1,0}) \right] Dy)|^2 dxdt \\
\leqslant& 2 \iint_Q \tilde{\sigma}^2 |\nabla (2\nu'_{1,0}(Dy, D\tilde{y}) - \frac{1}{\lambda}(\nu_{1,\lambda} - \nu_{1,0}))|^2 |\nabla \tilde{y}|^2 dxdt \\
&+ 2 \iint_Q \tilde{\sigma}^2 |2\nu'_{1,0}(Dy, D\tilde{y}) - \frac{1}{\lambda}(\nu_{1,\lambda} - \nu_{1,0})|^2 |\Delta \tilde{y}|^2 dxdt \\
=& 2 \iint_Q \tilde{\sigma}^2 |\nabla (2\nu'_{1,0}(Dy, D\tilde{y}) - 2\nu'_{1,\tilde{\lambda}(t)}(Dy, D\tilde{y}))|^2 |\nabla \tilde{y}|^2 dxdt \\
&+ 2 \iint_Q \tilde{\sigma}^2 |2\nu'_{1,0}(Dy, D\tilde{y}) - 2\nu'_{1,\tilde{\lambda}(t)}(Dy, D\tilde{y})|^2 |\Delta \tilde{y}|^2 dxdt \\
\leqslant& 8 \iint_Q \tilde{\sigma}^2 \|Dy\|^2 \|D\tilde{y}\|^2 |\nabla \nu'_{1,0} - \nabla \nu'_{1,\tilde{\lambda}(t)}|^2 |\nabla \tilde{y}|^2 dxdt \\
&+ 8 \iint_Q \tilde{\sigma}^2 \|Dy\|^2 \|D\tilde{y}\|^2 |\nu'_{1,0} - \nu'_{1,\tilde{\lambda}(t)}|^2 |\Delta \tilde{y}|^2 dxdt
\end{aligned}$$

where $0 < \tilde{\lambda}(t) \leqslant \lambda$.

Using the Lebesgue's Theorem, we can find that $B_2 \to 0$, as $\lambda \to 0$. Indeed

$$\tilde{\sigma}^2(t)\|Dy(t)\|^2\|D\tilde{y}(t)\|^2|\nabla \nu'_{1,0}(x,t) - \nabla \nu'_{1,\tilde{\lambda}(t)}(x,t)|^2|\nabla \tilde{y}(x,t)|^2 \to 0, \quad \text{as } \lambda \to 0,$$

$$\tilde{\sigma}^2(t)\|Dy(t)\|^2\|D\tilde{y}(t)\|^2|\nu'_{1,0}(x,t) - \nu'_{1,\tilde{\lambda}(t)}(x,t)|^2|\Delta \tilde{y}(x,t)|^2 \to 0, \quad \text{as } \lambda \to 0,$$

and

$$\tilde{\sigma}^2\|Dy\|^2\|D\tilde{y}\|^2|\nabla \nu'_{1,0} - \nabla \nu'_{1,\tilde{\lambda}(t)}|^2|\nabla \tilde{y}|^2 \leqslant C\kappa^6\|\nabla y\|^2\|\nabla \tilde{y}\|^2|\nabla \tilde{y}|^2 \in L^1(Q),$$

$$\tilde{\sigma}^2\|Dy\|^2\|D\tilde{y}\|^2|\nu'_{1,0} - \nu'_{1,\tilde{\lambda}(t)}|^2|\Delta \tilde{y}|^2 \leqslant C\kappa^6\|\nabla y\|^2\|\nabla \tilde{y}\|^2|\Delta \tilde{y}|^2 \in L^1(Q).$$

Estimative for B_3

$$\begin{aligned}
B_3 :=& \iint_Q \tilde{\sigma}^2 \lambda^2 |(\tilde{y} \cdot \nabla)\tilde{y}|^2 dxdt \\
&\leqslant C \int_0^T \xi \kappa^3 \lambda^2 \|(\tilde{y} \cdot \nabla)\tilde{y}\|^2 dt \\
&\leqslant \lambda^2 C \int_0^T \xi \kappa^3 \|\nabla \tilde{y}\|^3 \|\Delta \tilde{y}\| dt \\
&\leqslant \lambda^2 C \|(\tilde{y}, \tilde{p}, \tilde{v}, \tilde{\theta}, \tilde{v}_0)\|_{Y_N}^4
\end{aligned}$$

where $B_3 \to 0$, as $\lambda \to 0$.

On the other hand, we can prove analogously that

$$\|\frac{1}{\lambda}\left[\mathcal{A}_3((y,p,v,\theta,v_0)+\lambda(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0))-\mathcal{A}_3(y,p,v,\theta,v_0)\right]-D\mathcal{A}_3(y,p,v,\theta,v_0)\|^2_{L^2(\hat{\sigma}^2;Q)}$$

$$=\|\nabla\cdot((\nu_{1,0}-\nu_{1,\sigma})\nabla\tilde{\theta})+\nabla\cdot(\left[2\nu'_{1,0}(Dy,D\tilde{y})-\frac{1}{\lambda}(\nu_{1,\lambda}-\nu_{1,0})\right]\nabla\theta)+\lambda\tilde{y}\cdot\nabla\tilde{\theta}\|^2_{L^2(\hat{\sigma}^2;Q)}$$

$$\leqslant 3\|\nabla\cdot((\nu_{1,0}-\nu_{1,\sigma})\nabla\tilde{\theta})\|^2+3\|\nabla\cdot(\left[2\nu'_{1,0}(Dy,D\tilde{y})-\frac{1}{\lambda}(\nu_{1,\lambda}-\nu_{1,0})\right]\nabla\theta)\|^2+3\|\lambda\tilde{y}\cdot\nabla\tilde{\theta}\|^2$$

$$=3\tilde{B}_1+3\tilde{B}_2+3\tilde{B}_3$$

where $\tilde{B}_i\to 0$, for $1\leqslant i\leqslant 3$ as $\lambda\to 0$.

Thus, $\mathcal{A}$ is $G-$differentiable at any $(y,p,v,\theta,v_0)\in Y_N$, with

$$\mathcal{A}'_G(y,p,v,\theta,v_0)=D\mathcal{A}(y,p,v,\theta,v_0),\quad \forall(y,p,v,\theta,v_0)\in Y_N.$$

Now, we shall prove that the mapping $(y,p,v,\theta,v_0)\mapsto\mathcal{A}'_G(y,p,v,\theta,v_0)$ is continuous from Y_N into $\mathcal{L}(Y_N,Z_N)$.

Thus, let us assume that $(y_n,p_n,v_n,\theta_n,v_{0n})\overset{Y_N}{\to}(y,p,v,\theta,v_0)$ and let us check that

$$\|(D\mathcal{A}(y_n,p_n,v_n,\theta_n,v_{0n})-D\mathcal{A}(y,p,v,\theta,v_0))(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0)\|^2_{Z_N}\leqslant\epsilon_n\|(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0)\|^2_{Y_N} \tag{2.19}$$

for all $(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0)\in Y_N$, for some $\epsilon_n\to 0$.

First, we have that

$$\begin{aligned}
&\|(D\mathcal{A}_1(y_n,p_n,v_n,\theta_n,v_{0n})-D\mathcal{A}_1(y,p,v,\theta,v_0))(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0)\|^2_{L^2(\tilde{\sigma}^2;Q)}\\
=&\|(\tilde{y}_t-\nabla\cdot((\nu_0+\tilde{\nu}_{1,n})D\tilde{y})-2(Dy_n,D\tilde{y})\nabla\cdot(\tilde{\nu}'_{1,n}Dy_n)+\nabla\tilde{p}+(\tilde{y}\cdot\nabla)y_n+(y_n\cdot\nabla)\tilde{y}\\
&-\tilde{v}1_\omega-\tilde{\theta}e_N)-(\tilde{y}_t-\nabla\cdot((\nu_0+\nu_{1,0})D\tilde{y})-2(Dy,D\tilde{y})\nabla\cdot(\nu'_{1,0}Dy)+\nabla\tilde{p}\\
&+(\tilde{y}\cdot\nabla)y+(y\cdot\nabla)\tilde{y}-\tilde{v}1_\omega-\tilde{\theta}e_N)\|^2_{L^2(\tilde{\sigma}^2;Q)}\\
\leqslant&3\|\nabla\cdot((\nu_{1,0}-\tilde{\nu}_{1,n})D\tilde{y})\|^2_{L^2(\tilde{\sigma}^2;Q)}+12\|\nabla\cdot(\nu'_{1,0}(\nabla y,\nabla\tilde{y})Dy-\tilde{\nu}'_{1,n}(\nabla y_n,\nabla\tilde{y})Dy_n)\|^2_{L^2(\tilde{\sigma}^2;Q)}\\
&+3\|(\tilde{y}\cdot\nabla)(y_n-y)+((y_n-y)\cdot\nabla)\tilde{y}\|^2_{L^2(\tilde{\sigma}^2;Q)}\\
:=&3D_{1,n}+12D_{2,n}+3D_{3,n}.
\end{aligned}$$

Using again Proposition 2.3.5 and properties of ν_1, we obtain

$$\begin{aligned}
D_{1,n}&\leqslant 2\iint_Q\tilde{\sigma}^2|\nabla(\tilde{\nu}_{1,n}-\nu_{1,0})|^2|\nabla\tilde{y}|^2dxdt+2\iint_Q\hat{\sigma}^2|\tilde{\nu}_{1,n}-\nu_{1,0}|^2|\Delta\tilde{y}|^2dxdt\\
&\leqslant C\iint_Q\tilde{\sigma}^2|\|Dy_n\|^2-\|Dy\|^2|^2(|\nabla\tilde{y}|^2+|\Delta\tilde{y}|^2)dxdt\\
&\leqslant C\iint_Q\kappa^6\|\nabla(y_n-y)\|^2(\|\nabla y_n\|^2+\|\nabla y\|^2)(|\nabla\tilde{y}|^2+|\Delta\tilde{y}|^2)dxdt\\
&\leqslant\epsilon_{1,n}\|(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0)\|^2_{Y_N}
\end{aligned}$$

where

$$\epsilon_{1,n} := C\|(y_n-y, p_n-p, v_n-v, \theta_n-\theta, v_{0n}-v_0)\|^2(\|(y_n, p_n, v_n, \theta_n, v_{0n})\|^2+\|(y, p, v, \theta, v_0)\|^2).$$

Also

$$\begin{aligned}
D_{2,n} \leqslant& 3\iint_Q \tilde{\sigma}^2|\nabla.(\tilde{\nu}'_{1,n}(D(y_n-y), D\tilde{y})Dy_n)|^2 dxdt \\
&+ 3\iint_Q \tilde{\sigma}^2|\nabla.(\tilde{\nu}'_{1,n}(Dy, D\tilde{y})D(y_n-y))|^2 dxdt \\
&+ 3\iint_Q \tilde{\sigma}^2|\nabla.((\tilde{\nu}'_{1,n}-\nu'_{1,0})(Dy, D\tilde{y})Dy)|^2 dxdt \\
\leqslant& 3\iint_Q \tilde{\sigma}^2\|D(y_n-y)\|^2\|D\tilde{y}\|^2|\nabla.(\tilde{\nu}'_{1,n}Dy_n)|^2 dxdt \\
&+ 3\iint_Q \tilde{\sigma}^2\|Dy\|^2\|D\tilde{y}\|^2|\nabla.(\tilde{\nu}'_{1,n}D(y_n-y))|^2 dxdt \\
&+ 3\iint_Q \tilde{\sigma}^2\|Dy\|^2\|D\tilde{y}\|^2|\nabla.((\tilde{\nu}'_{1,n}-\nu'_{1,0})Dy)|^2 dxdt \\
\leqslant& C\left(\iint_Q \tilde{\sigma}^2\|\nabla(y_n-y)\|^2\|\nabla\tilde{y}\|^2(|\nabla\tilde{\nu}'_{1,n}|^2|\nabla y_n|^2+|\tilde{\nu}'_{1,n}|^2|\Delta y_n|^2)dxdt\right. \\
&+ \iint_Q \tilde{\sigma}^2\|\nabla y\|^2\|\nabla\tilde{y}\|^2(|\nabla\tilde{\nu}'_{1,n}|^2|\nabla(y_n-y)|^2+|\tilde{\nu}'_{1,n}|^2|\Delta(y_n-y)|^2)dxdt \\
&\left.+ \iint_Q \tilde{\sigma}^2\|\nabla y\|^2\|\nabla\tilde{y}\|^2(|\nabla\tilde{\nu}'_{1,n}-\nabla\nu'_{1,0}|^2|\nabla y_n|^2+|\tilde{\nu}'_{1,n}-\nu'_{1,0}|^2|\Delta y_n|^2)dxdt\right) \\
\leqslant& C\left(\iint_Q \kappa^6\|\nabla(y_n-y)\|^2\|\nabla\tilde{y}\|^2(|\nabla y_n|^2+|\Delta y_n|^2)dxdt\right. \\
&+ \iint_Q \kappa^6\|\nabla y\|^2\|\nabla\tilde{y}\|^2(|\nabla(y_n-y)|^2+|\Delta(y_n-y)|^2)dxdt \\
&\left.+ \iint_Q \kappa^6\|\nabla y\|^2\|\nabla\tilde{y}\|^2(|\nabla\tilde{\nu}'_{1,n}-\nabla\nu'_{1,0}|^2|\nabla y_n|^2+|\tilde{\nu}'_{1,n}-\nu'_{1,0}|^2|\Delta y_n|^2)dxdt\right) \\
\leqslant& C\left\{\sup_{[0,T]} \kappa^2\|\nabla\tilde{y}\|^2\right\}\left(\iint_Q \kappa^4\|\nabla(y_n-y)\|^2(|\nabla y_n|^2+|\Delta y_n|^2)dxdt\right. \\
&+ \iint_Q \kappa^4\|\nabla y\|^2(|\nabla(y_n-y)|^2+|\Delta(y_n-y)|^2)dxdt \\
&\left.+ \iint_Q \kappa^4\|\nabla y\|^2(|\nabla\tilde{\nu}'_{1,n}-\nabla\nu'_{1,0}|^2|\nabla y_n|^2+|\tilde{\nu}'_{1,n}-\nu'_{1,0}|^2|\Delta y_n|^2)dxdt\right) \\
\leqslant& \epsilon_{2,n}\|(\tilde{y}, \tilde{p}, \tilde{v}, \tilde{\theta}, \tilde{v}_0)\|^2_{Y_N}
\end{aligned}$$

where

$$\begin{aligned}
\epsilon_{2,n} :=& C\left(\|(y_n-y, p_n-p, v_n-v, \theta_n-\theta, v_{0n}-v_0)\|^2(\|(y_n, p_n, v_n, \theta_n, v_{0n})\|^2+\|(y, p, v, \theta, v_0)\|^2)\right. \\
&\left.+ \iint_Q \kappa^4\|\nabla y\|^2(|\nabla\tilde{\nu}'_{1,n}-\nabla\nu'_{1,0}|^2|\nabla y_n|^2+|\tilde{\nu}'_{1,n}-\nu'_{1,0}|^2|\Delta y_n|^2)dxdt\right).
\end{aligned}$$

And

$$\begin{aligned}
D_{3,n} &\leqslant 2\iint_Q \tilde{\sigma}^2(|(\tilde{y}.\nabla)(y_n-y)|^2+|((y_n-y).\nabla)\tilde{y}|^2)dxdt\\
&\leqslant C\int_0^T \tilde{\sigma}^2(\|\tilde{y}\|_{L^6}^2\|\nabla(y_n-y)\|_{L^3}^2+\|y_n-y\|_{L^6}^2\|\nabla\tilde{y}\|_{L^3}^2)dt\\
&\leqslant C\int_0^T \xi\kappa^3(\|\nabla\tilde{y}\|^2\|\nabla(y_n-y)\|\|\Delta(y_n-y)\|+\|\nabla(y_n-y)\|^2\|\nabla\tilde{y}\|\|\Delta\tilde{y}\|)dt\\
&\leqslant \epsilon_{3,n}\|(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0)\|_{Y_N}^2
\end{aligned}$$

where

$$\epsilon_{3,n} := C\|(y_n-y,p_n-p,v_n-v,\theta_n-\theta,v_{0n}-v_0)\|^2.$$

Using Lebesgue's Theorem, it easy see that $\epsilon_{i,n}\to 0$ as $n\to+\infty$ for $1\leqslant i\leqslant 3$.
And finally, we proceed analogously as $D\mathcal{A}_1$

$$\begin{aligned}
&\|(D\mathcal{A}_3(y_n,p_n,v_n,\theta_n,v_{0n})-D\mathcal{A}_3(y,p,v,\theta,v_0))(\tilde{y},\tilde{p},\tilde{v},\tilde{\theta},\tilde{v}_0)\|_{L^2(\hat{\sigma}^2;Q)}^2\\
=&\|(\tilde{\theta}_t-\nabla\cdot((\nu_0+\tilde{\nu}_{1,n})\nabla\tilde{\theta})-2(Dy_n,D\tilde{y})\nabla\cdot(\tilde{\nu}_{1,n}'\nabla\theta_n)-\tilde{v}_0 1_\omega+\tilde{y}\cdot\nabla\theta_n+y_n\cdot\nabla\tilde{\theta})\\
&-(\tilde{\theta}_t-\nabla\cdot((\nu_0+\nu_{1,0})\nabla\tilde{\theta})-2(Dy,D\tilde{y})\nabla\cdot(\nu_{1,0}'\nabla\theta)-\tilde{v}_0 1_\omega+\tilde{y}\cdot\nabla\theta+y\cdot\nabla\tilde{\theta})\|_{L^2(\hat{\sigma}^2;Q)}^2\\
\leqslant&3\|\nabla\cdot((\nu_{1,0}-\tilde{\nu}_{1,n})\nabla\tilde{\theta})\|_{L^2(\tilde{\sigma}^2;Q)}^2+12\|\nabla\cdot(\nu_{1,0}'(Dy,D\tilde{y})\nabla\theta-\tilde{\nu}_{1,n}'(Dy_n,D\tilde{y})\nabla\theta_n\|_{L^2(\tilde{\sigma}^2;Q)}^2\\
&+3\|\tilde{y}\cdot\nabla(\theta_n-\theta)+(y_n-y)\cdot\nabla\tilde{\theta}\|_{L^2(\hat{\sigma}^2;Q)}^2\\
:=&3\tilde{D}_{1,n}+12\tilde{D}_{2,n}+3\tilde{D}_{3,n}\\
\leqslant&\tilde{\epsilon}_n\|(y',p',v',\theta',v_0')\|_{Y_N}^2
\end{aligned}$$

where $\tilde{\epsilon}_n\to 0$ as $n\to+\infty$ for $1\leqslant i\leqslant 3$.
This shows that (2.19) is satisfied. □

Lemma 2.4.3. $\mathcal{A}'(0,0,0,0,0): Y_N\to Z_N$ *is onto.*

Proof. We know that,

$$\begin{aligned}
\mathcal{A}'(0,0,0,0,0)(y,p,v,\theta,v_0) = (&y_t+\nabla\cdot(\nu(t)Dy)+\nabla p-v1_\omega-\theta e_N, y(0),\\
&\theta_t-\nabla\cdot(\nu(t)\nabla\theta)-v_0 1_\omega,\theta(0)).
\end{aligned}$$

Let us fix $(f,y_0,f_0,\theta_0)\in Z_N$, from Proposition 2.3.3, we know that there exist (y,p,v,θ,v_0) satisfying $v_N\equiv 0$, $v_k\equiv 0$, $k<N$, $\tilde{\rho}y$, $\tilde{\eta}v1_\omega\in L^2(Q)^N$, $\tilde{\rho}\theta$, $\hat{\eta}v_0 1_\omega\in L^2(Q)$, $\tilde{\sigma}(y_t+\nabla\cdot(\nu(t)Dy)+\nabla p-v1_\omega-\theta e_N)\in L^2(Q)^N$, $\hat{\sigma}(\theta_t-\nabla\cdot(\nu(t)\nabla\theta)-v_0 1_\omega)\in$

$L^2(Q)$.

From the usual regularity results for the Stokes system, we have

$$y \in L^2(0,T;D(A)), \quad \theta \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \quad p \in L^2(0,T;H^1(\Omega)).$$

Thus $(y,p,v,\theta,v_0) \in Y_N$. $\square$

In accordance with Lemmas 2.4.1, 2.4.2 and 2.4.3, we can apply Theorem 1.1.2, thus, there exist $\epsilon > 0$ and a mapping $W : B_\epsilon(0) \subset Z_N \to Y_N$ such that

$$W(z) \in B_r(0) \text{ and } \mathcal{A}(W(z)) = z, \quad \forall z \in B_\epsilon(0).$$

Taking $(0,y_0,0,\theta_0) \in B_\epsilon(0)$ and $(y,p,v,\theta,v_0) = W(0,y_0,0,\theta_0) \in Y_N$, we have

$$\mathcal{A}((y,p,v,\theta,v_0)) = (0,y_0,0,\theta_0).$$

Therefore, (2.1) is locally null controllable at time $T > 0$.

2.5 Additional Commentary

For the case $\nabla\nu(0,x,t) \neq 0$, we can obtain similar results, this is, we can prove Theorem 2.1.1 only with $v_N \equiv 0$, in other words, we only get the local null controllability for the system (2.1) with N scalar controls.

For this, we must prove the Carleman Inequality of the same form as [4] in Proposition 2.1 with the additional term $\nu(x,t)$. Then we verify the additional estimates $(2.3.4)-(2.3.5)$ and finally the proof of Lemmas $2.4.1-2.4.2$.

Chapter 3

On the Theoretical Control of a 1D Nonlinear Parabolic PDE

3.1 Introduction

We will analyze the null controllability of the nonlinear systems

$$\begin{cases} y_t - (a(y)y_x)_x = v1_\omega & \text{in} \quad Q \\ y(0,t) = y(L,t) = 0 & \text{in} \quad (0,T) \\ y(x,0) = y_0(x) & \text{in} \quad I \end{cases} \tag{3.1}$$

and

$$\begin{cases} y_t - (a(y)y_x)_x = 0 & \text{in} \quad Q \\ y(0,t) = w(t), \ y(L,t) = 0 & \text{on} \quad (0,T) \\ y(x,0) = y_0(x) & \text{in} \quad I, \end{cases} \tag{3.2}$$

where v and w are the controls and y is in both cases the associated state. Here, it will be assumed that the real function $a = a(r)$ is of class $C^2(\mathbb{R})$ possesses bounded derivatives of order $\leqslant 2$ and satisfies

$$0 < m \leqslant a(r) \leqslant M \quad \forall r \in \mathbb{R}.$$

Definition 3.1.1. *It will be said that* (3.1) *(resp.* (3.2)*) is locally null-controllable at time* T *if there exists* $\epsilon > 0$ *such that, for any* $y_0 \in H_0^1(I)$ *with*

$$\|y_0\|_{H_0^1(I)} \leqslant \epsilon,$$

there exist controls $v \in L^2(\omega \times (0,T))$ *(resp.* $w \in L^2(0,T)$*) such that the associated states* y *satisfy*

$$y(x,T) = 0 \ \ in \ \ I. \tag{3.3}$$

The controllability of linear and semilinear parabolic PDEs and systems has been the objective of a lot of work the last decades; some relevant contributions on the subject are [11, 16, 18]. However, to our knowledge, nothing has been established up to now for systems like (3.1) or (3.2).

Our main result in this chapter is the following:

Theorem 3.1.2. *Under the previous assumptions on* $a(\cdot)$*, the nonlinear system* (3.1) *is locally null-controllable at any time* $T > 0$*.*

A consequence of Theorem 3.1.2 is the local null controllability of (3.2). Thus, our second result is the following:

Theorem 3.1.3. *Under the previous assumptions on* $a(\cdot)$*, the nonlinear system* (3.2) *is locally null-controllable at any time* $T > 0$*.*

The proof of Theorem 3.1.2 will be performed as described in Chapter 1.

Thus, in a first step, we consider the following linearized system at zero

$$\begin{cases} y_t - a(0)y_{xx} = v1_\omega + h & \text{in} \quad Q \\ y(0,t) = y(L,t) = 0 & \text{in} \quad (0,T) \\ y(x,0) = y_0(x) & \text{in} \quad I. \end{cases} \tag{3.4}$$

The adjoint of (3.4) is given by

$$\begin{cases} -\varphi_t - a(0)\varphi_{xx} = F & \text{in} \quad Q \\ \varphi(0,t) = \varphi(L,t) = 0 & \text{in} \quad (0,T) \\ \varphi(x,T) = \varphi_T(x), & \text{in} \quad I, \end{cases} \tag{3.5}$$

where $F \in L^2(Q)$ and $\varphi_T \in L^2(I)$. The null controllability of (3.4) will be obtained, for appropriate right hand sides h, as a consequence of a global Carleman inequality for the solutions to (3.5).

This chapter is organized as follows.

Section 3.2 is devoted to recall some known results and prove the null controllability of the linearized system (3.4).

In Section 3.3, we prove Theorems 3.1.2 and 3.1.3.

Section 3.4 deals with some additional comments and results.

3.2 Some Technical Results

3.2.1 Carleman Inequalities

In this section, we will recall some Carleman inequalities satisfied by the solutions to (3.5). They are well known consequences of the results in [18].

It will be convenient to introduce a new non-empty open set ω_0, with $\omega_0 \Subset \omega$.

Let us introduce the functions

$$\beta(t) := t(T-t), \quad \phi(x,t) := \frac{e^{\lambda\alpha_0(x)}}{\beta(t)}, \quad \alpha(x,t) := \frac{e^{R\lambda} - e^{\lambda\alpha_0(x)}}{\beta(t)},$$

where α_0 is obtained in Lemma 1.1.1 (with $\Omega := I$), $R > \|\alpha_0\|_{L^\infty}$ and $\lambda > 0$.

Then one has the following:

Proposition 3.2.1. *There exist positive constants λ_0, s_0 and C_0 such that, for any $s \geqslant s_0$ and $\lambda \geqslant \lambda_0$, any $F \in L^2(Q)$ and any $\varphi_T \in L^2(I)$, the associated solution to* (3.5) *satisfies*

$$\begin{aligned}\iint_Q e^{-2s\alpha}[(s\phi)^{-1}(|\varphi_t|^2 + |\varphi_{xx}|^2) + \lambda^2(s\phi)|\varphi_x|^2 + \lambda^4(s\phi)^3|\varphi|^2]\,dx\,dt \\ \leqslant C_0\left(\iint_Q e^{-2s\alpha}|F|^2\,dx\,dt + \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\lambda^4(s\phi)^3|\varphi|^2\,dx\,dt\right).\end{aligned} \tag{3.6}$$

Furthermore, C_0 and λ_0 (resp. s_0) only depend on I, ω and $a(0)$ (resp. the same and T).

The next result contains a Carleman inequality for the solution to (3.5) with weights not vanishing at zero. Let m be a function satisfying

$$m \in C^\infty([0,T]), \quad m(t) \geqslant \frac{T^2}{8} \quad \text{in} \quad [0,T/2], \quad m(t) = t(T-t) \quad \text{in} \quad [T/2,T],$$

let us set

$$\zeta(x,t) := \frac{e^{\lambda\alpha_0(x)}}{m(t)}, \quad A(x,t) := \frac{e^{R\lambda} - e^{\lambda\alpha_0(x)}}{m(t)} \quad \text{with} \quad R > \|\alpha_0\|_{L^\infty}, \quad \lambda > 0$$

and let us introduce the notation

$$\Gamma(s,\lambda,\varphi) := \iint_Q e^{-2sA}[(s\zeta)^{-1}(|\varphi_t|^2 + |\varphi_{xx}|^2) + \lambda^2(s\zeta)|\varphi_x|^2 + \lambda^4(s\zeta)^3|\varphi|^2]\,dx\,dt.$$

Proposition 3.2.2. *There exist positive constants λ_1, s_1 and C_1 such that, for any $s \geqslant s_1$ and $\lambda \geqslant \lambda_1$, any $F \in L^2(Q)$ and any $\varphi_T \in L^2(I)$, the associated solution to* (3.5) *satisfies*

$$\Gamma(s,\lambda,\varphi) \leqslant C_1(s,\lambda) \left(\iint_Q e^{-2sA}|F|^2\,dx\,dt + \iint_{\omega\times]0,T[} e^{-2sA}\zeta^3|\varphi|^2\,dx\,dt \right).$$

Furthermore, C_1, s_1 and λ_1 only depend on I, ω, $a(0)$ and T.

See the detailed proof of Proposition 3.2.2 in [8].

3.2.2 Null Controllability of (3.4)

In order to simplify the notation, we fix from now on $\lambda = \lambda_1$ and $s = s_1$ and we set

$$\rho_i := \zeta^{-i/2}e^{sA}, \quad i \in \mathbb{N}.$$

Due to Proposition 3.2.2, we will be able to prove the null controllability of (3.4) for right hand sides h that decay sufficiently fast to zero as $t \to T$. More precisely, one has the following:

Proposition 3.2.3. *Assume that the function h satisfies*

$$\iint_Q \rho_3^2|h|^2\,dx\,dt < +\infty.$$

Then (3.4) *is null controllable. More precisely, for any $y_0 \in L^2(I)$, there exist controls $v \in L^2(\omega\times(0,T))$ and associated states y satisfying*

$$\iint_{\omega\times(0,T)} \rho_3^2|v|^2\,dx\,dt < +\infty, \quad \int_Q \rho_0^2|y|^2\,dx\,dt < +\infty, \tag{3.7}$$

whence, in particular, $y(x,T) \equiv 0$.

The proof of this result is classical; see [18] for the details.

3.2.3 Some Estimates of the State

The next results provide additional properties of the state found in Proposition 3.2.3. They will be needed below, in Section 3.3.

Proposition 3.2.4. *Let the hypotheses in Proposition* 3.2.3 *be satisfied and let v and y satisfy* (3.4) *and* (3.7). *Then*

$$\begin{aligned}\sup_{[0,T]}\left(\int_I {\rho_5}^2|y|^2\,dx\right) + \iint_Q {\rho_5}^2|y_x|^2\,dx\,dt \leqslant C\Bigg(\iint_Q \rho_0^2|y|^2\,dx\,dt \\ + \iint_{\omega\times(0,T)} \rho_3^2|v|^2\,dx\,dt + \|y_0\|^2_{L^2(I)} + \iint_Q \rho_3^2|h|^2\,dx\,dt\Bigg).\end{aligned} \tag{3.8}$$

Proof. Multiplying (3.4) by ${\rho_5}^2 y$ and integrating in I, we get:

$$\int_I {\rho_5}^2(y_t - a(0)y_{xx})y\,dx = \int_I {\rho_5}^2(v1_\omega + h)y\,dx.$$

Note that, using integration by parts and Holder inequalities, we have

- $\displaystyle\int_I {\rho_5}^2 y_t y\,dx = \frac{1}{2}\frac{d}{dt}\left(\int_I {\rho_5}^2|y|^2\,dx\right) - \int_I \rho_5\rho_{5t}|y|^2\,dx,$
- $\displaystyle -\int_I {\rho_5}^2 a(0)y_{xx}y\,dx = -\frac{1}{2}\int_I a(0)({\rho_5}^2)_{xx}|y|^2\,dx + \int_I {\rho_5}^2 a(0)|y_x|^2\,dx,$
- $\displaystyle\int_I {\rho_5}^2 v1_\omega y\,dx \leqslant \frac{1}{2}\int_I {\rho_5}^2|y|^2\,dx + \frac{1}{2}\int_\omega {\rho_5}^2|v|^2\,dx,$
- $\displaystyle\int_I {\rho_5}^2(hy)\,dx \leqslant \frac{1}{2}\int_I {\rho_5}^4\rho_3^{-2}|y|^2\,dx + \frac{1}{2}\int_I \rho_3^2|h|^2\,dx.$

Therefore,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\int_I {\rho_5}^2|y|^2\,dx\right) + \int_I {\rho_5}^2 a(0)|y_x|^2 dx \\ &\quad\leqslant C\Bigg(\int_I({\rho_5}^2 + \rho_5|\rho_{5t}| + |({\rho_5}^2)_{xx}| + {\rho_5}^4\rho_3^{-2})|y|^2\,dx \\ &\quad\quad + \int_\omega {\rho_5}^2|v|^2\,dx + \int_I \rho_3^2|h|^2\,dx\Bigg)\end{aligned}$$

and we see that

$$\begin{aligned}&\frac{d}{dt}\left(\int_I {\rho_5}^2|y|^2\,dx\right) + \int_I {\rho_5}^2|y_x|^2 dx \\ &\leqslant C\left(\int_I \rho_0^2|y|^2\,dx + \frac{1}{2}\int_\omega \rho_3^2|v|^2\,dx + \frac{1}{2}\int_I \rho_3^2|h|^2\,dx\right).\end{aligned}$$

Now, integrating in time, we get the desired estimate. □

Proposition 3.2.5. *Let the hypotheses in Proposition* 3.2.3 *be satisfied, let v and y satisfy* (3.4) *and* (3.7) *and let us assume that*

$$y_0 \in H_0^1(I). \tag{3.9}$$

Then one has

$$\sup_{[0,T]}\left(\int_I \rho_7^2|y_x|^2\,dx\right)+\iint_Q \rho_7^2(|y_t|^2+|y_{xx}|^2)\,dx\,dt$$
$$\leqslant C\left(\iint_Q \rho_0^2|y|^2\,dx\,dt+\iint_{\omega\times(0,T)}\rho_3^2|v|^2\,dx\,dt\right. \tag{3.10}$$
$$\left.+\|y_0\|^2_{H_0^1(I)}+\iint_Q \rho_3^2|h|^2\,dx\,dt\right).$$

Proof. This time, let us multiply (3.4) by $\rho_7^2 y_t$ and let us integrate in I. We find:

$$\int_I \rho_7^2(y_t-a(0)y_{xx})y_t\,dx=\int_I \rho_7^2(v1_\omega+h)y_t\,dx. \tag{3.11}$$

Now,

- $\displaystyle\int_I \rho_7^2 v1_\omega y_t\,dx\leqslant \frac{1}{8}\int_I \rho_7^2|y_t|^2\,dx+2\int_\omega \rho_7^2|v|^2\,dx,$

- $\displaystyle\int_I \rho_7^2 h y_t\,dx\leqslant \frac{1}{8}\int_I \rho_7^2|y_t|^2\,dx+2\int_I \rho_7^2|h|^2\,dx.$

- Also,

$$-\int_I \rho_7^2 a(0)y_{xx}y_t\,dx=\frac{1}{2}\frac{d}{dt}\int_I \rho_7^2 a(0)|y_x|^2\,dx$$
$$-\frac{1}{2}\int_I(\rho_7^2)_t a(0)|y_x|^2\,dx+\int_I(\rho_7^2)_x a(0)y_xy_t\,dx. \tag{3.12}$$

Thus, from (3.11), we obtain that

$$\int_I \rho_7^2|y_t|^2\,dx+\frac{1}{2}\frac{d}{dt}\int_I \rho_7^2 a(0)|y_x|^2\,dx$$
$$\leqslant \frac{1}{2}\int_I(\rho_7^2)_t a(0)|y_x|^2\,dx-\int_I(\rho_7^2)_x a(0)y_xy_t\,dx$$
$$+\frac{1}{4}\int_I \rho_7^2|y_t|^2\,dx+2\int_\omega \rho_7^2|v|^2\,dx+2\int_I \rho_7^2|h|^2\,dx.$$

We also have

$$\frac{1}{2}\int_I(\rho_7^2)_t a(0)|y_x|^2\,dx-\int_I(\rho_7^2)_x a(0)y_xy_t\,dx$$
$$\leqslant C\left(\int_I[(\rho_7^2)_t+\hat{\rho}_0{}^2]|y_x|^2\,dx\right)+\frac{1}{8}\int_I \rho_7^2|y_t|^2\,dx.$$

Therefore,

$$\int_I \rho_7^2|y_t|^2\,dx+\frac{d}{dt}\int_I \rho_7^2|y_x|^2\,dx$$
$$\leqslant C\left(\int_I {\rho_5}^2|y_x|^2\,dx+\int_\omega \rho_3^2|v|^2\,dx+\int_I \rho_3^2|h|^2\,dx\right).$$

From the definition of the weight ρ_7, we have

$$\frac{1}{2}\int_I \rho_7^2|y_t|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_I \rho_7^2 a(0)|y_x|^2\,dx \leqslant C\left(\int_I \hat{\rho}_7^2|y_x|^2\,dx + \int_\omega \rho_7^2|v|\,dx + \int_I \rho_7^2|h|^2\,dx\right).$$

Integrating in time and recalling (3.9) and (3.8), we deduce the estimate

$$\sup_{[0,T]}\left(\int_I \rho_7^2|y_x|^2\,dx\right) + \iint_Q \rho_7^2|y_t|^2\,dx\,dt \leqslant C\left(\iint_Q \rho_0^2|y|^2\,dx\,dt + \iint_{\omega\times(0,T)} \rho_3^2|v|^2\,dx\,dt + \|y_0\|^2_{H_0^1(I)} + \iint_Q \rho_3^2|h|^2\,dx\,dt\right). \tag{3.13}$$

Now, let us multiply (3.4) by $-\rho_7^2 y_{xx}$ and let us integrate in I. We find that

$$\int_I \rho_7^2(y_t - a(0)y_{xx})(-y_{xx})\,dx = \int_I \rho_7^2(v1_\omega + h)(-y_{xx})\,dx.$$

In view of the identity (3.12), we see that

$$\frac{1}{2}\frac{d}{dt}\int_I \rho_7^2|y_x|^2\,dx + \int_I \rho_7^2 a(0)|y_{xx}|^2\,dx = \frac{1}{2}\int_I (\rho_7^2)_t|y_x|^2\,dx - \int_I (\rho_7^2)_x y_t y_x\,dx - \int_I \rho_7^2 v1_\omega y_{xx}\,dx - \int_I \rho_7^2 h y_{xx}\,dx.$$

We also have the estimates

- $$-\int_I (\rho_7^2)_x y_t y_x\,dx \leqslant C\left(\int_I [(\rho_7^2)_x]^2\rho_7^{-2}|y_x|^2\,dx + \int_I \rho_7^2|y_t|^2\,dx\right),$$

- $$-\int_I \rho_7^2 v1_\omega y_{xx}\,dx \leqslant \frac{2}{a(0)}\int_\omega \rho_7^2|v|^2\,dx + \frac{a(0)}{8}\int_I \rho_7^2|y_{xx}|^2\,dx,$$

- $$-\int_I \rho_7^2 h y_{xx}\,dx \leqslant C\int_I \rho_7^2|h|^2\,dx + \frac{a(0)}{8}\int_I \rho_7^2|y_{xx}|^2\,dx.$$

Consequently,

$$\frac{d}{dt}\int_I \rho_7^2|y_x|^2\,dx + \int_I \rho_7^2|y_{xx}|^2\,dx \leqslant C\left(\int_I {\rho_5}^2|y_x|^2\,dx + \int_I \rho_7^2|y_t|^2\,dx + \int_\omega \rho_3^2|v|^2\,dx + \int_I \rho_3^2|h|^2\,dx\right)$$

and, integrating in time and recalling (3.13), we finally deduce (3.10). □

3.3 Proofs of the Main Results

This section is devoted to prove local null controllability results for (3.1) and (3.2).

3.3.1 Proof of Theorem 3.1.2

Let us set

$$
\begin{aligned}
Y := \{ (y,v) : &\textstyle\iint_{\omega\times(0,T)} \rho_3^2|v|^2\,dx\,dt < +\infty,\ \iint_Q \rho_0^2|y|^2\,dx\,dt < +\infty,\\
&\textstyle\iint_Q \rho_3^2|y_t - a(0)y_{xx} - v1_\omega|^2\,dx\,dt < +\infty,\ \sup_{[0,T]}\int_I \rho_7^2|y_x|^2\,dx < +\infty,\\
&\textstyle\iint_Q \rho_7^2\,[|y_t|^2 + |y_{xx}|^2]\,dx\,dt < +\infty,\quad y=0 \ \text{ on } \ \Sigma\},
\end{aligned}
$$

$$
F := \{ g \in L^2(Q) : \iint_Q \rho_3^2|g|^2\,dx\,dt < +\infty \}
$$

and

$$
Z := F \times H_0^1(I).
$$

We will use the following norms in Y, F and Z:

$$
\begin{aligned}
\|(y,v)\|_Y^2 := &\iint_Q \rho_0^2|y|^2\,dx\,dt + \iint_{\omega\times(0,T)} \rho_3^2|v|^2\,dx\,dt\\
&+ \textstyle\iint_Q \rho_3^2|y_t - a(0)y_{xx} - v1_\omega|^2\,dx\,dt\\
&+ \sup_{[0,T]} \int_I \rho_7^2|y_x|^2dx + \iint_Q \rho_7^2\left[|y_t|^2 + |y_{xx}|^2\right]\,dx\,dt.
\end{aligned}
$$

$$
\|g\|_F^2 := \iint_Q \rho_3^2|g|^2\,dx\,dt.
$$

and

$$
\|(g,z)\|_Z^2 := \|g\|_F^2 + \|z\|_{H_0^1(I)}^2.
$$

Note that, if $(y,v)\in Y$, then $y\in C^0([0,T];H_0^1(I))$ and, also,

$$
\max_{[0,T]} \|y(\cdot\,,t)\|_{H_0^1(I)} \leqslant C\|(y,v)\|_Y\,.
$$

Let us consider the mapping $\mathcal{A} \mapsto Z$, with

$$
\mathcal{A}(y,v) = (y_t - (a(y)y_x)_x - v1_\omega, y(\cdot,0)). \tag{3.14}
$$

We will use Liusternik's Theorem to prove that there exists $\epsilon > 0$ such that, whenever $(h,y_0)\in Z$ and $\|(h,y_0)\|_Z \leqslant \epsilon$, then the equation (**??**) possesses at least one solution. In particular, this will show that (3.1) is locally null-controllable, with state-control pairs $(y,v)\in Y$.

Now, our goal is to prove that we can apply this result to the mapping $\mathcal{A}$ in (3.14). We will use following lemmas:

Lemma 3.3.1. *Let $\mathcal{A} : Y \mapsto Z$ be the mapping defined by* (3.14)*. Then $\mathcal{A}$ is well defined and continuous.*

Proof. Let us assume that $(y, v) \in Y$, let us set $\mathcal{A}(y, v) = (\mathcal{A}_1(y, v), \mathcal{A}_2(y, v))$ and let us see that $\mathcal{A}_1(y, v)$ and $\mathcal{A}_2(y, v)$ make sense and belong to F and $H_0^1(I)$, respectively.

One has:

$$
\begin{aligned}
\textstyle\iint_Q \rho_3^2 |\mathcal{A}_1(y, v)|^2 \, dx \, dt &= \iint_Q \rho_3^2 |y_t - (a(y)y_x)_x - v 1_\omega|^2 \, dx \, dt \\
&\leqslant C \textstyle\iint_Q \rho_3^2 |y_t - a(0)y_{xx} - v 1_\omega|^2 \, dx \, dt \\
&\quad + C \iint_Q \rho_3^2 |(a(y)y_x)_x - a(0)y_{xx}|^2 \, dx \, dt \\
&= I_1 + I_2.
\end{aligned}
$$

From the definition of Y, we see that

$$
I_1 = C \iint_Q \rho_3^2 |y_t - a(0)y_{xx} - v 1_\omega|^2 \, dx \, dt \leqslant C \|(y, v)\|_Y^2.
$$

On the other hand, since $a(\cdot) \in C^1(\mathbb{R})$ and is (globally) Lipschitz-continuous, one also has:

$$
\begin{aligned}
I_2 &= C \iint_Q \rho_3^2 |(a(y)y_x)_x - a(0)y_{xx}|^2 \, dx \, dt \\
&\leqslant C \iint_Q \rho_3^2 |a(y) - a(0)|^2 |y_{xx}|^2 \, dx \, dt + C \iint_Q \rho_3^2 |a'(y)|^2 |y_x|^4 \, dx \, dt \\
&\leqslant C \iint_Q \rho_3^2 |y|^2 |y_{xx}|^2 \, dx \, dt + C \iint_Q \rho_3^2 |y_x|^4 \, dx \, dt \\
&\leqslant C \left(\textstyle\sup_Q \rho_3^2 \rho_7^{-2} |y|^2\right) \iint_Q \rho_7^2 |y_{xx}|^2 \, dx \, dt + C \iint_Q \rho_3^2 |y_x|^4 \, dx \, dt.
\end{aligned}
$$

From the definitions of ρ_3, ρ_5 and ρ_7, we have $\rho_3^2 \rho_7^{-2} \leqslant \rho_7^2$ and $|(\rho_3 \rho_7^{-1})_x|^2 \leqslant {\rho_5}^2$ and, consequently,

$$
\begin{aligned}
\sup_{[0,T]} \left(\sup_I \rho_3^2 \rho_7^{-2} |y|^2 \right) &\leqslant \sup_{[0,T]} \left(\sup_I |\rho_3 \rho_7^{-1} y| \right)^2 \\
&\leqslant C \sup_{[0,T]} \int_I |(\rho_3 \rho_7^{-1} y)_x|^2 dx \\
&\leqslant C \sup_{[0,T]} \int_I (\rho_3^2 \rho_7^{-2} |y_x|^2 + |(\rho_3 \rho_7^{-1})_x|^2 |y|^2) dx \\
&\leqslant C \left(\sup_{[0,T]} \int_I \rho_7^2 |y_x|^2 dx + \sup_{[0,T]} \int_I {\rho_5}^2 |y|^2 dx \right) \\
&\leqslant C \|(y, v)\|_Y^2.
\end{aligned}
\tag{3.15}
$$

Moreover,

$$
\begin{aligned}
\iint_Q \rho_3^2 |y_x|^4 \, dx\, dt &\leqslant C \int_0^T \left(\sup_I \rho_3^2 \rho_7^{-2} |y_x|^2 \right) \left(\int_I \rho_7^2 |y_x|^2 \, dx \right) \, dt \\
&\leqslant C \left(\int_0^T \left(\sup_I \rho_3^2 \rho_7^{-2} |y_x|^2 \right) \, dt \right) \left(\sup_{[0,T]} \int_I \rho_7^2 |y_x|^2 \, dx \right) \\
&\leqslant C \left(\iint_Q \left(\rho_7^2 |y_{xx}|^2 + {\rho_5}^2 |y_x|^2 \right) dxdt \right) \left(\sup_{[0,T]} \int_I \rho_7^2 |y_x|^2 \, dx \right) \\
&\leqslant C \|(y,v)\|_Y^4.
\end{aligned}
\tag{3.16}
$$

Note that, in these inequalities, it is crucial that the spatial domain is one-dimensional, hence $H^1(I) \hookrightarrow L^\infty(I)$. Combining (3.15) and (3.16), the following is obtained:

$$
I_2 \leqslant C \|(y,v)\|_Y^4 \, .
$$

Therefore, $\mathcal{A}$ is well defined.

Furthermore, using similar arguments, it is easy to check that $\mathcal{A}$ is continuous. □

Lemma 3.3.2. *The mapping $\mathcal{A} \mapsto Z$ is continuously differentiable.*

Proof. Let us fix (y,v) in Y and let us choose arbitrary $(y',v') \in Y$ and $\sigma > 0$. We have:

$$
\begin{aligned}
\frac{1}{\sigma} \left[\mathcal{A}_1((y,v) + \sigma(y',v')) - \mathcal{A}_1(y,v) \right] = y'_t &- \frac{1}{\sigma} \left[a'(y+\sigma y')((y+\sigma y')_x^2 - y_x^2) \right] \\
&- \frac{1}{\sigma} \left[a'(y+\sigma y') - a'(y) \right] y_x^2 - a(y+\sigma y') y'_{xx} \\
&- \frac{1}{\sigma} \left[a(y+\sigma y') - a(y) \right] y_{xx} - v' 1_\omega .
\end{aligned}
$$

Let us consider the linear mapping $D\mathcal{A} : Y \mapsto Z$ given by

$$
\begin{aligned}
&D\mathcal{A} = (D\mathcal{A}_1, D\mathcal{A}_2). \\
&D\mathcal{A}_1(y',v') := y'_t - 2a'(y) y_x y'_x - a''(y) y' y_x^2 - a(y) y'_{xx} - a'(y) y' y_{xx} - v' 1_\omega , \\
&D\mathcal{A}_2(y',v') := y'(\cdot, 0).
\end{aligned}
$$

We claim that

$$
\frac{1}{\sigma} [\mathcal{A}_1((y,v) - \sigma(y',v')) - \mathcal{A}_1(y,v)] \to D\mathcal{A}_1(y',v') \ \text{ strongly in } \ F \tag{3.17}
$$

as $\sigma \to 0$.

Indeed,

$$
\begin{aligned}
&\left\|\frac{1}{\sigma}[\mathcal{A}_1((y,v)+\sigma(y',v'))-\mathcal{A}_1(y,v)]-D\mathcal{A}_1(y',v')\right\|_F \\
&\qquad\leqslant \|2a'(y)y'_x y_x - \frac{1}{\sigma}\left[a'(y+\sigma y')((y+\sigma y')_x^2-y_x^2)\right]\|_F \\
&\qquad\quad + \|a''(y)y'y_x^2 - \frac{1}{\sigma}\left[a'(y+\sigma y')-a'(y)\right]y_x^2\|_F \\
&\qquad\quad + \|a'(y)y'y_{xx} - \frac{1}{\sigma}\left[a(y+\sigma y')-a(y)\right]y_{xx}\|_F \\
&\qquad\quad + \|a(y)y'_{xx} - a(y+\sigma y')y'_{xx}\|_F \\
&\qquad = B_1+B_2+B_3+B_4.
\end{aligned}
$$

Let us check that the $B_i \to 0$ as $\sigma \to 0$. First, one has

$$
B_1^2 = \iint_Q \rho_3^2|2a'(y)y_x y'_x - a'(y+\sigma y')(2y'_x y_x - \sigma y'_x)|^2\,dx\,dt \to 0,
$$

as a consequence of Lebesgue's Theorem and the fact that $a(\cdot) \in C^1(\mathbb{R})$.

Let us denote by a''_* and a'_{**} the derivatives of a at some intermediate points. Using now that $a(\cdot) \in C^2(\mathbb{R})$ and, again, Lebesgue's Theorem, we have:

$$
\begin{aligned}
B_2^2 &= \iint_Q \rho_3^2|a''(y)y'y_x^2 - \frac{1}{\sigma}[a'(y+\sigma y')-a'(y)]y_x^2|^2\,dx\,dt \\
&= \textstyle\iint_Q \rho_3^2|a''(y)-a''_*|^2|y'y_x^2|^2\,dx\,dt \to 0
\end{aligned}
$$

and

$$
\begin{aligned}
B_3^2 &= \iint_Q \rho_3^2(a'(y)y'y_{xx} - \frac{1}{\sigma}[a(y+\sigma y')-a(y)]y_{xx})^2\,dx\,dt \\
&= \textstyle\iint_Q \rho_3^2((a'(y)-a'_{**})y'y_{xx})^2\,dx\,dt \to 0.
\end{aligned}
$$

A similar argument shows that B_4^2 also converges to zero as $\sigma \to 0$. Thus, (3.17) holds.

Let us denote by $\mathcal{A}'(y,v)$ the linear mapping $D\mathcal{A}$. It is clear that $\mathcal{A}'(y,v) \in \mathcal{L}(Y;Z)$. Let us prove that $(y,v) \mapsto \mathcal{A}'(y,v)$ is a continuous mapping. This will be sufficient to achieve the proof.

Thus, let us assume that $(y^n,v^n) \to (y,v)$ in Y and let us check that

$$
\|(D\mathcal{A}(y^n,v^n) - D\mathcal{A}(y,v))(y',v')\|_Z \leqslant \epsilon_n\|(y',v')\|_Y \text{ for some } \epsilon_n \to 0. \tag{3.18}
$$

Observe that

$$
\begin{aligned}
\|(D\mathcal{A}_1(y^n,v^n) - D\mathcal{A}_1(y,v))(y',v')\|_F^2 \\
\leqslant C \iint_Q \rho_3^2 |a'(y^n)y_x^n y_x' - a'(y)y_x y_x'|^2 \,dx\,dt \\
+ C \iint_Q \rho_3^2 |a''(y^n)y'(y_x^n)^2 - a''(y)y'y_x^2|^2 \,dx\,dt \\
+ C \iint_Q \rho_3^2 |a(y^n)y_{xx}' - a(y)y_{xx}'|^2 \,dx\,dt \\
+ C \iint_Q \rho_3^2 |a'(y^n)y'y_{xx}^n - a'(y)y'y_{xx}|^2 \,dx\,dt \\
= D_1 + D_2 + D_3 + D_4.
\end{aligned}
$$

Then, after some tedious but straightforward computations, we see that

$$
D_1 \leqslant C\|(y^n,v^n) - (y,v)\|_Y^2 (1 + \|(y,v)\|_Y^2)\|(y',v')\|_Y^2,
$$

$$
D_2 \leqslant C\|(y^n,v^n) - (y,v)\|_Y^2 (1 + \|(y,v)\|_Y^2)\|(y,v)\|_Y^2\|(y',v')\|_Y^2
$$

and similar estimates hold for D_3 and D_4.

Accordingly, (3.18) is satisfied and the proof is done. □

Lemma 3.3.3. *Let $\mathcal{A}$ be the mapping defined by (3.14). Then $\mathcal{A}'(0,0) \in \mathcal{L}(Y;Z)$ is onto.*

Proof. Let us consider the linear mapping $\mathcal{A}'(0,0) = (K_1, K_2)$. We have

$$
\begin{cases}
K_1(y',v') = y_t' - a(0)y_{xx}' - v'1_\omega \\
K_2(y',v') = y'(\cdot,0)
\end{cases}
\tag{3.19}
$$

for all $(y',v') \in Y$. Observe that $\mathcal{A}'(0,0)$ is onto if and only if for each $(g,y_0) \in Z$ there exist $(y,v) \in Y$ satisfying

$$
\begin{cases}
y_t - a(0)y_{xx} = v1_\omega + g & \text{in} \quad Q \\
y(0,t) = y(L,t) = 0 & \text{in} \quad (0,T) \\
y(x,0) = y_0(x) & \text{in} \quad I.
\end{cases}
$$

From Proposition 3.2.3 and Proposition 3.2.5, there exists a couple (y,v) with the desired properties. Consequently, the lemma holds. □

From the previous lemmas, we see that, in the present context, all the assumptions in Theorem 1.1.2 are satisfied. Thus, this result can be applied, (3.1) is locally null-controllable and Theorem 3.1.2 holds.

3.3.2 Proof of Theorem 3.1.3

Let us set $I_\delta = (-\delta, L)$ with $\delta > 0$ and let $\tilde{\omega} \subset I_\delta \setminus \overline{I}$ be a non-empty open set.

Let us consider the following auxiliary system:

$$\begin{cases} \tilde{y}_t - (a(\tilde{y})\tilde{y}_x)_x = \tilde{v}1_{\tilde{\omega}} & \text{in} \quad I_\delta \times (0,T) \\ \tilde{y}(-\delta, t) = \tilde{y}(L,t) = 0 & \text{in} \quad (0,T) \\ \tilde{y}(x,0) = \tilde{y}_0(x) & \text{in} \quad I_\delta, \end{cases} \tag{3.20}$$

where $\tilde{y}_0 \in H_0^1(I_\delta)$ is the extension-by-zero of y_0 to I_δ.

From Theorem 3.1.2, we deduce the existence of a control $\tilde{v} \in L^2(\tilde{\omega} \times (0,T))$ and an associated state $\tilde{y}$ solving (3.20) and satisfying

$$\tilde{y}(x,T) = 0 \quad \text{in} \quad I_\delta.$$

Let w be the trace of $\tilde{y}$ on $\partial I \times (0,T)$. Then, the couple (y, w), where y is the restriction of $\tilde{y}$ to $I \times (0,T)$, solves the corresponding system (3.2).

This proves Theorem 3.1.3.

3.4 Some Additional Results

3.4.1 Controllability to Trajectories

It is possible to find control results to trajectories, similar to the Theorems 3.1.2 and 3.1.3.

Define the uncontrollable trajectory

$$\begin{cases} \overline{y}_t - (a(\overline{y})\overline{y}_x)_x = 0 & \text{in} \quad Q, \\ \overline{y}(0,t) = \overline{y}(L,t) = 0 & \text{in} \quad (0,T), \\ \overline{y}(x,0) = \overline{y}_0(x) & \text{in} \quad I \end{cases} \tag{3.21}$$

where $\overline{y}_0 \in H_0^1(I) \cap H^3(I)$. The following holds:

Theorem 3.4.1. *Assume that $a(\cdot)$ is defined as in Theorem 3.1.2, with a'' is globally Lipschitz, then (3.1) (resp. (3.2)) is local exact controllable to trajectories at time $T > 0$, this is, there exists $\epsilon > 0$ such that, if y_0 is a state satisfying $y_0 \in H_0^1(I) \cap H^3(I)$ and*

$$\|y_0 - \overline{y}_0\| \leqslant \epsilon$$

there exist controls $v \in L^2(\omega \times (0,T))$ *(resp.* $v \in L^2(0,T)$*) and associated states* y *satisfying*

$$y(x,T) = \overline{y}(x,T) \quad \textit{in} \quad I.$$

3.4.2 Other Nonlinear Control Problem

The local null controllability of the system

$$\begin{cases} y_t - (a(y_x)y_x)_x = v1_\omega, & \text{in} \quad Q, \\ y(0,t) = y(L,t) = 0 & \text{in} \quad (0,T), \\ y(x,0) = y_0, & \text{in} \quad I \end{cases} \tag{3.22}$$

is possible, but using another ideas that we will show in the following chapter.

Chapter 4

Stackelberg-Nash Controllability of a Nonlinear Parabolic Equation

4.1 Introduction

We are interested in proving the exact controllability to trajectories of a multi-objective parabolic PDE problem in Q, where we apply the Stackelberg-Nash strategy; we will assume without less of generality that only three controls are applied (one leader and two followers).

We will consider the following system

$$\begin{cases} y_t - (a(y_x)y_x)_x + F(y, y_x) = f1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} \text{ in } Q, \\ y(0,t) = y(L,t) = 0 \quad \text{in} \ \ (0,T), \\ y(0) = y_0 \ \ \text{in} \ \ I. \end{cases} \tag{4.1}$$

In system (4.1), y is the state, the set $\mathcal{O} \subset I$ is the main control domain and $\mathcal{O}_1, \mathcal{O}_2 \subset I$ are the secondary control domain (it is supposed to be small); the controls are f, v^1 and v^2, where f is the leader and v^1, v^2 are the followers.

We assume that $a \in C^3(\mathbb{R})$, there exist positive constants a_0, a_1 such that $a_0 \leqslant a(s) \leqslant a_1, \forall s \in \mathbb{R}$, there exists a positive constant M such that $|a'(s)| + |a''(s)| + |a'''(s)| \leqslant M, \forall s \in \mathbb{R}$ and $a'(0) = 0$. $F \in C^2(\mathbb{R} \times \mathbb{R})$ with bounded derivatives and $F(0,0) = 0$.

Let $\mathcal{O}_{1,d}, \mathcal{O}_{2,d} \subset I$ be open sets, representing observation domains for the followers.

We will consider the functional

$$J_i(f; v^1, v^2) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} |y - y_{i,d}|^2 dxdt + \frac{\mu_i}{2} \iint_{\mathcal{O}_i\times(0,T)} |v^i|^2 dxdt, \tag{4.2}$$

where $\alpha_i, \mu_i > 0$ are constants and $y_{i,d} \in L^2(\mathcal{O}_{i,d} \times (0,T))$ are given function.
The control process can be described as follows:
1. The followers v^i assume that the leader f has made a choice and intend to be a Nash equilibrium for the costs J_i. Thus, once f has been fixed, we look for controls $v^i \in L^2(\mathcal{O}_i \times (0,T))$ that satisfy

$$J_1(f; v^1, v^2) = \min_{\hat{v}^1} J_1(f; \hat{v}^1, v^2), \quad J_2(f; v^1, v^2) = \min_{\hat{v}^2} J_2(f; v^1, \hat{v}^2). \tag{4.3}$$

Definition 4.1.1. *Any pair* (v^1, v^2) *satisfying* (4.3) *is called a Nash equilibrium for* J_1 *and* J_2*.*

Note that, if the functional J_i ($i = 1, 2$) are convex, then (v^1, v^2) is a Nash equilibrium if and only if

$$J_1'(f; v^1, v^2)(\hat{v}^1, 0) = 0, \quad \forall \hat{v}^1 \in L^2(\mathcal{O}_1 \times (0,T)) \tag{4.4}$$

and

$$J_2'(f; v^1, v^2)(0, \hat{v}^2) = 0, \quad \forall \hat{v}^2 \in L^2(\mathcal{O}_2 \times (0,T)). \tag{4.5}$$

Definition 4.1.2. *Any pair* (v^1, v^2) *satisfying* (4.4) *and* (4.5) *is called a Nash quasi-equilibrium for* J_1 *and* J_2*.*

2. Once the Nash equilibrium has been identified and fixed for each f, we look for a control $\hat{f} \in L^2(\mathcal{O} \times (0,T))$ subject to the restriction of null controllability

$$y(T) = 0 \text{ in } I. \tag{4.6}$$

4.1.1 The Main Results

Let us study the following problems.

Theorem 4.1.3. *Let us assume that*

$$\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset, \quad i = 1, 2 \tag{4.7}$$

Also, suppose that one of the following two conditions hold

$$\mathcal{O}_{1,d} = \mathcal{O}_{2,d} \tag{4.8}$$

or

$$\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}. \tag{4.9}$$

Then, there exist $\epsilon > 0$, $\mu_0 > 0$ *only depending on* I, T, $\mathcal{O}$, $\mathcal{O}_i$, $\mathcal{O}_{i,d}$ *and* α_i *and a positive function* $\hat{\rho} = \hat{\rho}(t)$ *blowing up at* $t = T$ *with the following property: if* $\mu \geqslant \mu_0$, *the* $y_{i,d}$ *is such that*

$$\iint_{\mathcal{O}_{i,d}\times(0,T)} \hat{\rho}^2 |y_{i,d}|^2 dxdt < \epsilon, \tag{4.10}$$

there exists $\delta > 0$, *such that for any* $y_0 \in H^3(I) \cap H_0^1(I)$ *with* $\|y_0\|_{H^3(I)\cap H_0^1(I)} \leqslant \delta$, *there exist controls* $f \in L^2(\mathcal{O} \times (0,T))$ *and associated Nash quasi-equilibrium* (v^1, v^2) *such that the corresponding solutions to* (4.1) *satisfy* (4.6)*.*

A natural question is whether there are semilinear systems for which the concepts of Nash equilibrium and Nash quasi-equilibrium are equivalent. An answer is given by the following result:

Theorem 4.1.4. *Let us assume that* $y_{i,d} \in L^\infty(\mathcal{O}_{i,d} \times (0,T))$. *Suppose that* $y_0 \in H^3(I) \cap H_0^1(I)$ *with* $\|y_0\|_{H^3(I)\cap H_0^1(I)} \leqslant \delta$. *Then, there exists* $C > 0$ *such that, if* $f \in L^2(\mathcal{O} \times (0,T))$ *and* μ *satisfies*

$$\mu \geqslant C(1 + \|y^0\|_{H_0^1(I)} + \|f\|_{L^2(\mathcal{O}\times(0,T))}),$$

the pair (v^1, v^2) *is a Nash equilibrium for* J_i *of* (4.1)*.*

The rest of the chapter is organized as follows. In Section 2 we prove Theorem 4.1.3, which concerns the Nash quasi-equilibrium with restriction of exact controllability to trajectories using Carleman inequalities and Right Inverse Function Theorem for Banach spaces. In section 3 we prove Theorem 4.1.4, which concerns the Nash equilibrium using techniques of [3].

4.2 Nash Quasi-Equilibrium for (4.1)

In this section we prove Theorem 4.1.3. The proof is divided in four steps: first, we perform a change of variable that reduces the task to solve a null controllability problem; second we will make the characterization of Nash quasi-equilibrium for (4.1); third we will study the exact controllability to trajectories of linearized system for this characterization using Carleman inequalities; finally in the fourth step, we will conclude the proof using Inverse Function Theorem for Banach spaces.

4.2.1 Characterization of Nash Quasi-Equilibrium

Note that the convexity of the functional J_i are not guaranteed. For this reason, we must re-define the concept of Nash optimally (recall Def. 4.1.2).

Note that (4.4)-(4.5) is equivalent to

$$\begin{cases} \alpha_i \displaystyle\iint_{\mathcal{O}_{i,d}\times(0,T)} (y - y_{i,d})\hat{y}^i \; dxdt + \mu_i \iint_{\mathcal{O}_i\times(0,T)} v^i \hat{v}^i \; dxdt = 0, \\ \forall \hat{v}^i \in L^2(\mathcal{O}_i \times (0,T)), \quad v^i \in L^2(\mathcal{O}_i \times (0,T)), \quad i = 1, 2, \end{cases} \tag{4.11}$$

where we have denoted by $\hat{y}^i$ the derivative of the state y with respect to v^i in the direction $\hat{v}^i$. One has

$$\begin{cases} \hat{y}^i_t - ((a'(y_x)y_x + a(y_x))\hat{y^i}_x)_x + D_1F(y,y_x)\hat{y}^i + D_2F(y,y_x)\hat{y}^i_x = \hat{v}^i 1_{\mathcal{O}} \text{ in } Q, \\ \hat{y}^i(0,t) = \hat{y}^i(L,t) = 0 \quad \text{in} \quad (0,T), \\ \hat{y}^i(0) = 0 \quad \text{in} \quad I. \end{cases} \tag{4.12}$$

Let us introduce the adjoint systems for (4.12)

$$\begin{cases} -p^i_t - ((a'(y_x)y_x + a(y_x))p^i_x)_x + D_1F(y,y_x)p^i - (D_2F(y,y_x)p^i)_x = \alpha_i(y - y_{i,d})1_{\mathcal{O}_{i,d}} \text{ in } Q, \\ p^i(0,t) = p^i(L,t) = 0 \quad \text{in} \quad (0,T), \\ p^i(T) = 0 \quad \text{in} \quad I. \end{cases} \tag{4.13}$$

If we multiply $(4.13)_1$ by $\hat{y}^i$ in $L^2(Q)$, and perform integration by parts, we obtain

$$\alpha_i \iint_Q (y - y_{i,d})1_{\mathcal{O}_{i,d}}\hat{y}^i \; dxdt = \iint_Q p^i \hat{v}^i \; 1_{\mathcal{O}_i} \; dxdt.$$

Replacing the above expression in (4.11), we have

$$\iint_Q p^i \hat{v}^i 1_{\mathcal{O}_i} \; dxdt + \mu_i \iint_{\mathcal{O}_i\times(0,T)} v^i \hat{v}^i \; dxdt = 0.$$

As a consequence, we get the following characterization of any Nash quasi-equilibrium for J_i

$$v^i = -\frac{1}{\mu_i} p^i 1_{\mathcal{O}_i}. \tag{4.14}$$

In this way, we have the following optimality systems for (4.1)

$$\begin{cases} y_t - (a(y_x)y_x)_x + F(y,y_x) = f1_{\mathcal{O}} - \dfrac{1}{\mu_1}p^1 1_{\mathcal{O}_1} - \dfrac{1}{\mu_2}p^2 1_{\mathcal{O}_2} \text{ in } Q, \\ -p^i_t - ((a'(y_x)y_x + a(y_x))p^i_x)_x + D_1F(y,y_x)p^i - (D_2F(y,y_x)p^i)_x = \alpha_i(y - y_{i,d})1_{\mathcal{O}_{i,d}} \text{ in } Q, \\ y(0,t) = y(L,t) = 0, \quad p^i(0,t) = p^i(L,t) = 0 \quad \text{in} \quad (0,T), \\ y(0) = y_0, \quad p^i(T) = 0 \quad \text{in} \quad I. \end{cases} \tag{4.15}$$

We consider the linearized system for (4.15)

$$\begin{cases} y_t - a(0)y_{xx} + D_1F(0,0)y + D_2F(0,0)y_x = f1_{\mathcal{O}} - \dfrac{1}{\mu_1}p^1 1_{\mathcal{O}_1} - \dfrac{1}{\mu_2}p^2 1_{\mathcal{O}_2} + G \text{ in } Q, \\ -p^i_t - a(0)p^i_{xx} + D_1F(0,0)p^i - D_2F(0,0)p^i_x = \alpha_i y 1_{\mathcal{O}_{i,d}} + G_i \text{ in } Q, \\ y(0,t) = y(L,t) = 0, \quad p^i(0,t) = p^i(L,t) = 0 \quad \text{in} \quad (0,T), \\ y(0) = y_0, \quad p^i(T) = 0, \quad \text{in} \quad I. \end{cases} \tag{4.16}$$

Now, we consider the adjoint system for (4.16)

$$\begin{cases} -\varphi_t - a(0)\varphi_{xx} + D_1F(0,0)\varphi - D_2F(0,0)\varphi_x = \alpha_1\theta^1 1_{\mathcal{O}_{1,d}} + \alpha_2\theta^2 1_{\mathcal{O}_{2,d}} + \mathcal{G} \text{ in } Q, \\ \gamma^i_t - a(0)\gamma^i_{xx} + D_1F(0,0)\gamma^i + D_2F(0,0)\gamma^i_x = -\dfrac{1}{\mu_i}\varphi 1_{\mathcal{O}_i} + \mathcal{G}_i \text{ in } Q, \\ \varphi(0,t) = \varphi(L,t) = 0, \quad \gamma^i(0,t) = \gamma^i(L,t) = 0, \quad \text{in} \quad (0,T), \\ \varphi(T) = \varphi^T, \quad \gamma^i(0) = 0, \quad \text{in} \quad I. \end{cases} \tag{4.17}$$

4.2.2 Null Controllability for (4.16)

In this section we prove the null controllability to the linearized system, in this way, we will need to define weight functions.

Let us consider a non-empty open set $\tilde{\mathcal{O}} \subset\subset \mathcal{O}$ such that $\mathcal{O}_{i,d} \cap \tilde{\mathcal{O}} \neq \emptyset$ for $i = 1,2$.

If (4.7) is satisfied, we can define $\mathcal{O}_d := \mathcal{O}_{1,d} = \mathcal{O}_{2,d}$ and we introduce the non-empty open set ω_0 satisfying $\omega_0 \subset\subset \mathcal{O}_d \cap \tilde{\mathcal{O}}$.

Let η^0 be the function obtained by Lemma 1.1.1 (with $\Omega := I$).

If (4.8) is satisfied, we introduce the non-empty connected open sets ω_i with

$$\omega_i \subset\subset \mathcal{O}_{i,d} \cap \tilde{\mathcal{O}}, \ i = 1,2 \ \ \omega_1 \cap \omega_2 \neq \emptyset. \tag{4.18}$$

such that

Lemma 4.2.1. *There exists functions* $\eta_i = \eta_i(x) \in C^2(\overline{I})$ $(i = 1, 2)$ *satisfying*

$$\begin{cases} \eta_i > 0, \ \text{in } I, \quad \eta_i = 0 \ \text{on } \partial I, \\ |\eta_{i,x}| > 0 \ \text{in } \ \overline{I}\backslash\omega_i, \quad \eta_1 = \eta_2 \ \text{in } I\backslash\tilde{\mathcal{O}}. \end{cases}$$

Proof. See [2]. □

Observation 4.2.2. *Lemma* 4.2.1 *establishes the existence of functions* η_1 *and* η_2 *which coincide outside* $\tilde{\mathcal{O}}$ *but may be very different inside* $\tilde{\mathcal{O}}$*. Nevertheless, it will be seen in the proof that one can find* η_1 *and* η_2 *satisfying* $\|\eta_1\|_\infty = \|\eta_2\|_\infty$.

Observation 4.2.3. *From* (4.7), (4.9) *and* (4.18), *we see that it can be assumed that either*

$$\omega_1 \cap \mathcal{O}_{2,d} = \emptyset \quad \text{and} \quad \omega_2 \cap \mathcal{O}_{1,d} = \emptyset \tag{4.19}$$

or

$$\omega_i \subset \mathcal{O}_{j,d} \ \ \text{and} \ \ \omega_j \cap \mathcal{O}_{i,d} = \emptyset, \quad \text{with} \ \ (i,j) = (1,2) \ \text{or} \ (i,j) = (2,1). \tag{4.20}$$

Let us introduce the weight functions

$$\sigma(x,t) := \frac{e^{4\lambda\|\eta^0\|_{L^\infty(\Omega)}} - e^{\lambda(2\|\eta^0\|_{L^\infty(\Omega)}+\eta^0(x))}}{t(T-t)}, \quad \xi(x,t) := \frac{e^{\lambda(2\|\eta^0\|_{L^\infty(\Omega)}+\eta^0(x))}}{t(T-t)},$$

$$\sigma_i(x,t) := \frac{e^{4\lambda\|\eta_i\|_{L^\infty(\Omega)}} - e^{\lambda(2\|\eta_i\|_{L^\infty(\Omega)}+\eta_i(x))}}{t(T-t)}, \quad \xi_i(x,t) := \frac{e^{\lambda(2\|\eta_i\|_{L^\infty(\Omega)}+\eta_i(x))}}{t(T-t)},$$

and the notations

$$I_m(\psi) := s^{m-4}\lambda^{m-3} \iint_Q e^{-2s\sigma}(\xi)^{m-4}(|\psi_t|^2 + |\psi_{xx}|^2)dxdt + L_m(\psi), \tag{4.21}$$

$$I^i_m(\psi) := s^{m-4}\lambda^{m-3} \iint_Q e^{-2s\sigma_i}(\xi_i)^{m-4}(|\psi_t|^2 + |\psi_{xx}|^2)dxdt + L^i_m(\psi), \tag{4.22}$$

where

$$L_m(\psi) := s^{m-2}\lambda^{m-1} \iint_Q e^{-2s\sigma}(\xi)^{m-2}|\psi_x|^2dxdt + s^m\lambda^{m+1} \iint_Q e^{-2s\sigma}(\xi)^m|\psi|^2dxdt,$$

$$L^i_m(\psi) := s^{m-2}\lambda^{m-1} \iint_Q e^{-2s\sigma_i}(\xi_i)^{m-2}|\psi_x|^2dxdt + s^m\lambda^{m+1} \iint_Q e^{-2s\sigma_i}(\xi_i)^m|\psi|^2dxdt.$$

Proposition 4.2.4. *Assume that* (4.7) − (4.9) *are satisfied. Then, there exists* $C(I, \mathcal{O}) > 0$ *such that, for every* $s \geqslant C(T + T^2)$ *and every* $\lambda \geqslant C$*, the solution* $(\varphi, \gamma^1, \gamma^2)$ *to* (4.17) *associated to* $\varphi^T \in L^2(I)$ *satisfies the following:*

i) If (4.8) *holds, then*

$$I_0(\varphi)+I_0(h)\leqslant C\left(s^{-3}\lambda^{-2}\iint_Q e^{-2s\sigma}(|\mathcal{G}|^2+|\mathcal{G}_1|^2+|\mathcal{G}_2|^2)dxdt\right.$$
$$\left.+\,s^4\lambda^5\iint_{\mathcal{O}\times(0,T)} e^{-2s\sigma}\xi^4|\varphi|^2dxdt\right). \tag{4.23}$$

ii) If (4.19) *holds, then*

$$I_0^1(\gamma^1)+I_0^2(\gamma^2)+s^{-3}\lambda^{-2}\iint_Q e^{-2s\sigma_1}(\xi_1)^{-3}|\varphi|^2dxdt$$
$$\leqslant C\left(s^{-3}\lambda^{-3}\iint_Q(e^{-2s\sigma_1}+e^{-2s\sigma_2})(|\mathcal{G}|^2+|\mathcal{G}_1|^2+|\mathcal{G}_2|^2)dxdt\right.$$
$$\left.+\,s^4\lambda^5\iint_{\mathcal{O}\times(0,T)}(e^{-2s\sigma_1}(\xi_1)^4+e^{-2s\sigma_2}(\xi_2)^4)|\varphi|^2dxdt\right). \tag{4.24}$$

iii) If (4.20) *holds for* $(i,j)=(i_0,j_0)$ *with* $(i_0,j_0)=(1,2)$ *or* $(i_0,j_0)=(2,1)$ *,then*

$$I_0^{j_0}(\gamma^{j_0})+I_0^{i_0}(h)+s^{-3}\lambda^{-2}\iint_Q e^{-2s\sigma_{j_0}}(\xi_{j_0})^{-3}|\varphi|^2dxdt$$
$$\leqslant C\left(s^{-3}\lambda^{-3}\iint_Q(e^{-2s\sigma_1}+e^{-2s\sigma_2})(|\mathcal{G}|^2+|\mathcal{G}_1|^2+|\mathcal{G}_2|^2)dxdt\right.$$
$$\left.+\,s^4\lambda^5\iint_{\mathcal{O}\times(0,T)}(e^{-2s\sigma_1}(\xi_1)^4+e^{-2s\sigma_2}(\xi_2)^4)|\varphi|^2dxdt\right). \tag{4.25}$$

where $h:=\alpha_1\gamma^1+\alpha_2\gamma^2$.

Proof. For *i)* see [3], and for *ii)* and *iii)* see [2]. □

We will apply a standard observability argument, in fact let us consider the following weight functions

$$l(t):=\begin{cases} T^2/4, \text{ for } 0\leqslant t\leqslant T/2,\\ t(T-t), \text{ for } T/2\leqslant t\leqslant T,\end{cases}$$

and

$$\overline{\sigma}(x,t):=\frac{e^{4\lambda\|\eta^0\|_{L^\infty(\Omega)}}-e^{\lambda(2\|\eta^0\|_{L^\infty(\Omega)}+\eta^0(x))}}{l(t)},\quad \overline{\xi}(x,t):=\frac{e^{\lambda(2\|\eta^0\|_{L^\infty(\Omega)}+\eta^0(x))}}{l(t)},$$

$$\overline{\sigma}_i(x,t):=\frac{e^{4\lambda\|\eta_i\|_{L^\infty(\Omega)}}-e^{\lambda(2\|\eta_i\|_{L^\infty(\Omega)}+\eta_i(x))}}{l(t)},\quad \overline{\xi}_i(x,t):=\frac{e^{\lambda(2\|\eta_i\|_{L^\infty(\Omega)}+\eta_i(x))}}{l(t)}.$$

We consider

$$\sigma^*(t) := \max_{x\in\Omega} \overline{\sigma}(x,t), \quad \hat{\sigma}(t) := \min_{x\in\Omega} \overline{\sigma}(x,t), \quad \xi^*(t) := \max_{x\in\Omega} \overline{\xi}(x,t),$$

$$\sigma_i^*(t) := \max_{x\in\Omega} \overline{\sigma}_i(x,t), \quad \hat{\sigma}_i(t) := \min_{x\in\Omega} \overline{\sigma}_i(x,t), \quad \xi_i^*(t) := \max_{x\in\Omega} \overline{\xi}_i(x,t).$$

If $\lambda > 1/\|\eta^0\|_\infty$ and $\lambda > 1/\|\eta_i\|_\infty$ (sufficiently large), we have

$$\hat{\sigma} \leqslant \overline{\sigma} < \frac{5}{4}\hat{\sigma}, \quad \frac{4}{5}\sigma^* < \overline{\sigma} \leqslant \sigma^*, \tag{4.26}$$

$$\hat{\sigma}_i \leqslant \overline{\sigma}_i < \frac{5}{4}\hat{\sigma}_i, \quad \frac{4}{5}\sigma_i^* < \overline{\sigma}_i \leqslant \sigma_i^*. \tag{4.27}$$

Let us denote by $\overline{I}_m(\varphi)$ the right-hand side of (4.21) with σ and ξ respectively replaced by $\overline{\sigma}$ and $\overline{\xi}$. Then, one can directly see from the energy estimate and the Proposition 4.2.4 that

i) If (4.8) holds, then

$$\begin{aligned}\|\varphi(0)\|^2 + \overline{I}_0(\varphi) + \overline{I}_0(h) \leqslant & C \left(s^{-3}\lambda^{-2} \iint_Q e^{-2s\overline{\sigma}}(|\mathcal{G}|^2 + |\mathcal{G}_1|^2 + |\mathcal{G}_2|^2) dxdt \right. \\ & \left. + s^4\lambda^5 \iint_{\mathcal{O}\times(0,T)} e^{-2s\overline{\sigma}}\overline{\xi}^4 |\varphi|^2 dxdt \right).\end{aligned}$$

ii) If (4.19) holds, then

$$\begin{aligned}\|\varphi(0)\|^2 + & s^{-3}\lambda^{-2} \iint_Q e^{-2s\overline{\sigma}_1}(\overline{\xi}_1)^{-3}|\varphi|^2 dxdt \\ \leqslant & C \left(s^{-3}\lambda^{-3} \iint_Q (e^{-2s\overline{\sigma}_1} + e^{-2s\overline{\sigma}_2})(|\mathcal{G}|^2 + |\mathcal{G}_1|^2 + |\mathcal{G}_2|^2) dxdt \right. \\ & \left. + s^4\lambda^5 \iint_{\mathcal{O}\times(0,T)} (e^{-2s\overline{\sigma}_1}(\overline{\xi}_1)^4 + e^{-2s\overline{\sigma}_2}(\overline{\xi}_2)^4)|\varphi|^2 dxdt \right).\end{aligned}$$

iii) If (4.20) holds for $(i,j) = (i_0, j_0)$ with $(i_0, j_0) = (1,2)$ or $(i_0, j_0) = (2,1)$,then

$$\begin{aligned}\|\varphi(0)\|^2 + & s^{-3}\lambda^{-2} \iint_Q e^{-2s\overline{\sigma}_{j_0}}(\tilde{\xi}_{j_0})^{-3}|\varphi|^2 dxdt \\ \leqslant & C \left(s^{-3}\lambda^{-3} \iint_Q (e^{-2s\overline{\sigma}_1} + e^{-2s\overline{\sigma}_2})(|\mathcal{G}|^2 + |\mathcal{G}_1|^2 + |\mathcal{G}_2|^2) dxdt \right. \\ & \left. + s^4\lambda^5 \iint_{\mathcal{O}\times(0,T)} (e^{-2s\overline{\sigma}_1}(\overline{\xi}_1)^4 + e^{-2s\overline{\sigma}_2}(\overline{\xi}_2)^4)|\varphi|^2 dxdt \right).\end{aligned}$$

Now, we denote

$$\beta(x,t) := \frac{2}{5}\overline{\sigma}(x,t), \quad \beta^*(t) := \max_{x\in\Omega} \beta(x,t), \quad \hat{\beta}(t) := \min_{x\in\Omega} \beta(x,t) \tag{4.28}$$

and

$$\beta_i(x,t) := \frac{2}{5}\overline{\sigma_i}(x,t), \quad \beta_i^*(t) := \max_{x\in\Omega} \beta_i(x,t), \quad \hat{\beta}_i(t) := \min_{x\in\Omega} \beta_i(x,t). \tag{4.29}$$

Using (4.26) – (4.29) in the last result, we get

i) If (4.8) holds, then

$$\|\varphi(0)\|^2 + \iint_Q e^{-5s\beta^*}|\varphi|^2 dxdt \leqslant C\left(\iint_Q e^{-4s\beta^*}(|\mathcal{G}|^2 + |\mathcal{G}_1|^2 + |\mathcal{G}_2|^2)dxdt + \iint_{\mathcal{O}\times(0,T)} e^{-4s\beta^*}(\xi^*)^4|\varphi|^2 dxdt\right).$$

ii) If (4.19) holds, then

$$\|\varphi(0)\|^2 + \iint_Q e^{-5s\beta_1^*}(\xi_1^*)^{-3}|\varphi|^2 dxdt \leqslant C\left(\iint_Q e^{-4s\beta_1^*}(|\mathcal{G}|^2 + |\mathcal{G}_1|^2 + |\mathcal{G}_2|^2)dxdt + \iint_{\mathcal{O}\times(0,T)} e^{-4s\beta_1^*}(\xi_1^*)^4|\varphi|^2 dxdt\right).$$

iii) If (4.20) holds for $(i,j) = (i_0,j_0)$ with $(i_0,j_0) = (1,2)$ or $(i_0,j_0) = (2,1)$,then

$$\|\varphi(0)\|^2 + \iint_Q e^{-5s\beta_{j_0}^*}(\xi_{j_0}^*)^{-3}|\varphi|^2 dxdt \leqslant C\left(\iint_Q e^{-4s\beta_1^*}(|\mathcal{G}|^2 + |\mathcal{G}_1|^2 + |\mathcal{G}_2|^2)dxdt + \iint_{\mathcal{O}\times(0,T)} e^{-4s\beta_1^*}(\xi_1^*)^4|\varphi|^2 dxdt\right).$$

Taking the PDE satisfied by the γ^i in (4.17), multiplying by $e^{-5s\beta}\gamma^i$ or $e^{-5s\beta_j}(\xi_j^*)^{-3}\gamma^i$, we easily see that

$$\iint_Q e^{-5s\beta^*}|\gamma^i|^2 dxdt \leqslant C\left(\iint_Q e^{-5s\beta^*}(|\mathcal{G}|^2 + |\mathcal{G}_1|^2 + |\mathcal{G}_2|^2)dxdt + \iint_Q e^{-5s\beta^*}|\varphi|^2 dxdt\right)$$

or

$$\iint_Q e^{-5s\beta_j^*}(\xi_j^*)^{-3}|\gamma^i|^2 dxdt \leqslant C\left(\iint_Q e^{-5s\beta_j^*}(\xi_j^*)^{-3}(|\mathcal{G}|^2 + |\mathcal{G}_1|^2 + |\mathcal{G}_2|^2)dxdt + \iint_Q e^{-5s\beta_j^*}(\xi_j^*)^{-3}|\varphi|^2 dxdt\right)$$

Then, joined the last results we obtain

$$\|\varphi(0)\|^2 + \iint_Q e^{-5s\beta^*}|\varphi|^2 dxdt + \iint_Q e^{-5s\beta^*}(|\gamma^1|^2 + |\gamma^2|^2)dxdt \leqslant C\left(\iint_Q e^{-4s\beta^*}(|\mathcal{G}|^2 + |\mathcal{G}_1|^2 + |\mathcal{G}_2|^2)dxdt + \iint_{\mathcal{O}\times(0,T)} e^{-4s\beta^*}(\xi^*)^4|\varphi|^2 dxdt\right).$$

or

$$\|\varphi(0)\|^2+\iint_Q e^{-5s\beta_1^*}(\xi_1^*)^{-3}|\varphi|^2dxdt+\iint_Q e^{-5s\beta_1^*}(\xi_1^*)^{-3}(|\gamma^1|^2+|\gamma^2|^2)dxdt$$
$$\leqslant C\left(\iint_Q e^{-4s\beta_1^*}(|\mathcal{G}|^2+|\mathcal{G}_1|^2+|\mathcal{G}_2|^2)dxdt+\iint_{\mathcal{O}\times(0,T)} e^{-4s\beta_1^*}(\xi_1^*)^4|\varphi|^2dxdt\right).$$

Finally, for the two cases, we have the new observability inequality

$$\|\varphi(0)\|^2+\iint_Q e^{-5s\overline{\beta}^*}(\overline{\xi}^*)^{-3}(|\varphi|^2+|\gamma^1|^2+|\gamma^2|^2)dxdt$$
$$\leqslant C\left(\iint_Q e^{-4s\overline{\beta}^*}(|\mathcal{G}|^2+|\mathcal{G}_1|^2+|\mathcal{G}_2|^2)dxdt+\iint_{\mathcal{O}\times(0,T)} e^{-4s\overline{\beta}^*}(\overline{\xi}^*)^4|\varphi|^2dxdt\right). \tag{4.30}$$

where $(\overline{\beta}^*,\overline{\xi}^*):=(\beta^*,\xi^*)$ or (β_1^*,ξ_1^*).

Let us define

$$\rho:=e^{5s\overline{\beta}^*/2}(\overline{\xi}^*)^{3/2},\quad \rho_0:=e^{2s\overline{\beta}^*},\quad \rho_1:=e^{2s\overline{\beta}^*}(\overline{\xi}^*)^{-2},$$
$$\rho_2:=e^{3s\overline{\beta}^*/2}(\overline{\xi}^*)^{-3},\quad \rho_3:=e^{3s\overline{\beta}^*/2}(\overline{\xi}^*)^{-8},\quad \rho_4=e^{3s\overline{\beta}^*/2}(\overline{\xi}^*)^{-9},\quad \rho_5:=e^{3s\overline{\beta}^*/2}(\overline{\xi}^*)^{-10}. \tag{4.31}$$

Proposition 4.2.5. *Assume that $\rho G\in L^2(Q)$, $\rho_3 G_t\in L^2(Q)$, $\rho G_i\in L^2(Q)$ and $G(0)\in H_0^1(I)$ $(i=1,2)$. Then (4.16) is null-controllable. More precisely, for any $y_0\in H^3(I)\cap H_0^1(I)$, there exists a control-state (y,p^1,p^2,f) satisfying*

$$f\in L^2(\mathcal{O}\times(0,T)),\quad y,p^1,p^2\in L^\infty(0,T;H_0^1(I))\cap L^2(0,T;H^2(I)) \tag{4.32}$$

such that

$$\iint_Q \rho_0^2(|y|^2+|p^1|^2+|p^2|^2)dxdt<+\infty,\quad \iint_{\mathcal{O}\times(0,T)}(\rho_1^2|f|^2+{\rho_3}^2|f_t|^2)dxdt<+\infty. \tag{4.33}$$

In particular $y(T)=0$.

Proof. Let us denote $Lw=w_t-a(0)w_{xx}+D_1F(0,0)w+D_2F(0,0)w_x$ and $L^*w=-w_t-a(0)w_{xx}+D_1F(0,0)w-D_2F(0,0)w_x$, then, we define a vectorial space

$$\mathcal{X}_0:=\{(u,z^1,z^2)\in C^2(\overline{I})^3; u=0,\ z^1=z^2=0\ \text{ on } \Sigma,\ \ z^1(0)=z^2(0)=0\}.$$

and an application $b : \mathcal{X}_0 \times \mathcal{X}_0 \to \mathbb{R}$

$$
\begin{aligned}
b((u,&z^1,z^2),(\tilde{u},\tilde{z},\tilde{z}^1,\tilde{z}^2)) \\
&:= \iint_Q \rho_0^{-2}(L^*u-\alpha_1 z^1 1_{\mathcal{O}_{1,d}}-\alpha_2 z^2 1_{\mathcal{O}_{2,d}})(L^*\tilde{u}-\alpha_1\tilde{z}^1 1_{\mathcal{O}_{1,d}}-\alpha_2\tilde{z}^2 1_{\mathcal{O}_{2,d}})dxdt \\
&+\sum_{i=1}^{2}\iint_Q \rho_0^{-2}(Lz^i+\frac{1}{\mu_i}u1_{\mathcal{O}_i})(L\tilde{z}^i+\frac{1}{\mu_i}\tilde{u}1_{\mathcal{O}_i})dxdt \\
&+\iint_{\mathcal{O}\times(0,T)} \rho_1^{-2}\ u\tilde{u}\ dxdt, \qquad \forall (u,z^1,z^2),(\tilde{u},\tilde{z}^1,\tilde{z}^2)\in\mathcal{X}_0.
\end{aligned}
$$

We will prove that $b(\cdot,\cdot)$ defines an inner product, for that, it is enough to prove: If $b((u,z^1,z^2),(u,z^1,z^2))=0$, then $(u,z^1,z^2)=(0,0,0)$. Indeed, we have

$$
\begin{aligned}
&\iint_Q \rho_0^{-2}|L^*u-\alpha_1 z^1 1_{\mathcal{O}_{1,d}}-\alpha_2 z^2 1_{\mathcal{O}_{2,d}}|^2 dxdt \\
&\qquad +\sum_{i=1}^{2}\iint_Q \rho_0^{-2}|Lz^i+\frac{1}{\mu_i}u1_{\mathcal{O}_i}|^2dxdt+\iint_{\mathcal{O}\times(0,T)}\rho_1^{-2}|u|^2dxdt=0.
\end{aligned}
$$

Thus, we obtain the system

$$
\left|
\begin{array}{l}
L^*u = 0+\alpha_1 z^1 1_{\mathcal{O}_{1,d}}+\alpha_2 z^2 1_{\mathcal{O}_{2,d}} \text{ in } Q, \\
Lz^i = 0-\dfrac{1}{\mu_i}u1_{\mathcal{O}_i} \text{ in } Q, \\
u(0,t)=u(L,t)=0, \ \ z^i(0,t)=z^i(L,t)=0, \quad \text{in } \ (0,T), \\
u(T)=u^T, \ \ z^i(0)=0, \quad \text{in } \ I.
\end{array}
\right. \tag{4.34}
$$

For the Proposition 4.2.4 on the system (4.34), we have

$$
\begin{aligned}
\|u(0)\|^2 +& \iint_Q e^{-5s\overline{\beta}^*}(\overline{\xi}^*)^{-3}(|u|^2+|z^1|^2+|z^2|)dxdt \\
\leqslant & C\Bigg(\iint_Q \rho_0^{-2}(|L^*u-\alpha_1 z^1 1_{\mathcal{O}_{1,d}}-\alpha_2 z^2 1_{\mathcal{O}_{2,d}}|^2 \\
& +|Lz^1+\frac{1}{\mu_1}u1_{\mathcal{O}_1}|^2+|Lz^2+\frac{1}{\mu_2}u1_{\mathcal{O}_2}|^2)dxdt \\
& +\iint_{\mathcal{O}\times(0,T)}\rho_1^{-2}|\varphi|^2dxdt\Bigg)=0.
\end{aligned}
$$

Then $(u,z^1,z^2)=(0,0,0)$. This proves that $b(\cdot,\cdot)$ define a inner product in $\mathcal{X}_0$. Now, let us define $\mathcal{X}$ the completion of $\mathcal{X}_0$ with this inner product, then $\mathcal{X}$ is a Banach space with norm induced by the inner product $b(\cdot,\cdot)$. Clearly $b(\cdot,\cdot)$ is a bilinear, symmetric, continuous and coercive application in $\mathcal{X}$.

Let us define the functional linear $\mathbb{G} : \mathcal{X} \to \mathbb{R}$ as

$$
< \mathbb{G},(u,z^1,z^2) > := (y_0,u(0)) + \iint_Q (Gu+G_1z^1+G_2z^2)dxdt.
$$

Let us see that $\mathbb{G}$ is continuous. Indeed, if $(u, z^1, z^2) \in \mathcal{X}$, we have

$$
\begin{aligned}
| < \mathbb{G}, (u, z^1, z^2) > | \leqslant & |(y_0, u(0))| + \iint_Q (||G|u||G_1||z^1| + |G_2||z^2|) dxdt \\
\leqslant & \|y_0\| \|u(0)\| + \left\{ \iint_Q \rho^2 (|G|^2 + |G_1|^2 + |G_2|^2) dxdt \right\}^{\frac{1}{2}} \cdot \\
& \left\{ \iint_Q \rho^{-2} (|u|^2 + |z^1|^2 + |z^2|^2) dxdt \right\}^{\frac{1}{2}} \\
\leqslant & \left\{ \|y_0\|^2 + \iint_Q \rho^2 (|G|^2 + |G_1|^2 + |G_2|^2) dxdt \right\}^{\frac{1}{2}} \\
& \left\{ \|u(0)\|^2 + \iint_Q \rho^{-2} (|u|^2 + |z^1|^2 + |z^2|^2) dxdt \right\}^{\frac{1}{2}} \\
\leqslant & C\ b((u, z^1, z^2), (u, z^1, z^2))^{\frac{1}{2}} = C\ \|(u, z^1, z^2)\|_{\mathcal{X}}.
\end{aligned}
$$

Then, for Lax-Milgram's Theorem, there exists an unique $(\hat{u}, \hat{z}^1, \hat{z}^2) \in \mathcal{X}$ such that

$$
b((\hat{u}, \hat{z}^1, \hat{z}^2), (u, z^1, z^2)) =< \mathbb{G}, (u, z^1, z^2) >, \quad \forall (u, z^1, z^2) \in \mathcal{X}. \tag{4.35}
$$

In other words,

$$
\begin{aligned}
& \iint_Q \rho_0^{-2} (L^* \hat{u} - \alpha_1 \hat{z}^1 1_{\mathcal{O}_{1,d}} - \alpha_2 \hat{z}^2 1_{\mathcal{O}_{2,d}}) (L^* u - \alpha_1 z^1 1_{\mathcal{O}_{1,d}} - \alpha_2 z^2 1_{\mathcal{O}_{2,d}}) dxdt \\
& + \sum_{i=1}^{2} \iint_Q \rho_0^{-2} (L \hat{z}^i + \frac{1}{\mu_i} \hat{u} 1_{\mathcal{O}_i}) (L z^i + \frac{1}{\mu_i} u 1_{\mathcal{O}_i}) dxdt \\
& + \iint_{\mathcal{O} \times (0,T)} \rho_1^{-2}\ \hat{u} u\ dxdt \\
& = (y_0, u(0)) + \iint_Q (G u + G_1 z^1 + G_2 z^2) dxdt.
\end{aligned} \tag{4.36}
$$

As $(\hat{u}, \hat{z}^1, \hat{z}^2) \in \mathcal{X}$, then

$$
\begin{cases}
\rho_0^{-1} (L^* \hat{u} - \alpha_1 \hat{z}^1 1_{\mathcal{O}_{1,d}} - \alpha_2 \hat{z}^2 1_{\mathcal{O}_{2,d}}) \in L^2(Q), \\
\rho_0^{-1} (L \hat{z}^i + \dfrac{1}{\mu_i} \hat{u} 1_{\mathcal{O}_i}) \in L^2(Q), \\
\rho_1^{-1} \hat{u} \in L^2(Q).
\end{cases}
$$

We define

$$
\begin{cases}
\hat{y} := \rho_0^{-2} (L^* \hat{u} - \alpha_1 \hat{z}^1 1_{\mathcal{O}_{1,d}} - \alpha_2 \hat{z}^2 1_{\mathcal{O}_{2,d}}) & \text{in } \ Q, \\
\hat{p}^i := \rho_0^{-2} (L \hat{z}^i + \dfrac{1}{\mu_i} \hat{u} 1_{\mathcal{O}_i}) & \text{in } \ Q, \\
\hat{f} := -\rho_1^{-2}\ \hat{u} & \text{in } \ \mathcal{O} \times (0,T).
\end{cases} \tag{4.37}
$$

Replacing (4.37) in (4.36), we obtain

$$\begin{aligned}
&\iint_Q \hat{y}(L^*u - \alpha_1 z^1 1_{\mathcal{O}_{1,d}} - \alpha_2 z^2 1_{\mathcal{O}_{2,d}})dxdt + \sum_{i=1}^{2} \iint_Q \hat{p}^i (Lz^i + \frac{1}{\mu_i} u 1_{\mathcal{O}_i})dxdt \\
=&(y_0, u(0)) + \iint_{\mathcal{O}\times(0,T)} \hat{y}u \; dxdt + \iint_Q (GuG_1 z^1 + G_2 z^2)dxdt
\end{aligned}$$

this is,

$$\begin{aligned}
&\iint_Q \hat{y}bdxdt + \sum_{i=1}^{2} \iint_Q \hat{p}^i b_i dxdt \\
=&(y_0, u(0)) + \iint_{\mathcal{O}\times(0,T)} \hat{y}u \; dxdt + \iint_Q (Gu + G_1 z^1 + G_2 z^2)dxdt
\end{aligned}$$

where (u, z^1, z^2) is solution of the system

$$\left|\begin{array}{l}
L^*u = b + \alpha_1 z^1 1_{\mathcal{O}_{1,d}} + \alpha_2 z^2 1_{\mathcal{O}_{2,d}} \text{ in } Q, \\
Lz^i = b_i - \dfrac{1}{\mu_i} u 1_{\mathcal{O}_i} \text{ in } Q, \\
u(0,t) = u(L,t) = 0, \quad z^i(0,t) = z^i(L,t) = 0, \quad \text{in } (0,T), \\
u(T) = 0, \quad z^i(0) = 0, \quad \text{in } I.
\end{array}\right.$$

Thus, $(\hat{y}, \hat{p}^1, \hat{p}^2)$ is a solution by transposition of the problem

$$\left|\begin{array}{l}
L\hat{y} = G + \hat{f} 1_{\mathcal{O}} - \dfrac{1}{\mu_1} p^1 1_{\mathcal{O}_1} - \dfrac{1}{\mu_2} p^2 1_{\mathcal{O}_2} \text{ in } Q, \\
L^*\hat{p}^i = G_i + \alpha_i \hat{y} 1_{\mathcal{O}_{i,d}} \text{ in } Q, \\
\hat{y}(0,t) = \hat{y}(L,t) = 0, \quad \hat{p}^i(0,t) = \hat{p}^i(L,t) = 0, \quad \text{in } (0,T), \\
\hat{y}(0) = y_0, \quad \hat{p}^i(T) = 0, \quad \text{in } I.
\end{array}\right. \tag{4.38}$$

Since G, G_1, G_2 are regular, using energy estimates, we have

$$\hat{y}, \; \hat{p}^1, \hat{p}^2 \in L^\infty(0,T; H_0^1(I)) \cap L^2(0,T; H^2(I)).$$

Also,

$$\begin{aligned}
\iint_Q \rho_0^2 |\hat{y}|^2 dxdt &= \iint_Q \rho_0^2 \rho_0^{-4} |L^*\hat{u} - \alpha_1 \hat{z}^1 1_{\mathcal{O}_{1,d}} - \alpha_2 \hat{z}^2 1_{\mathcal{O}_{2,d}}|^2 dxdt \\
&= \iint_Q \rho_0^{-2} |L^*\hat{u} - \alpha_1 \hat{z}^1 1_{\mathcal{O}_{1,d}} - \alpha_2 \hat{z}^2 1_{\mathcal{O}_{2,d}}|^2 dxdt < +\infty, \\
\iint_Q \rho_0^2 |\hat{p}^i|^2 dxdt &= \iint_Q \rho_0^2 \rho_0^{-4} |L\hat{z}^i + \frac{1}{\mu_i} \hat{u} 1_{\mathcal{O}_i}|^2 dxdt \\
&= \iint_Q \rho_0^{-2} |L\hat{z}^i + \frac{1}{\mu_i} \hat{u} 1_{\mathcal{O}_i}|^2 dxdt < +\infty, \\
\iint_{\mathcal{O}\times(0,T)} \rho_1^2 |\hat{f}|^2 dxdt &= \iint_{\mathcal{O}\times(0,T)} \rho_1^2 \rho_1^{-4} |\hat{u}|^2 dxdt = \iint_{\mathcal{O}\times(0,T)} \rho_1^{-2} |\hat{u}|^2 dxdt < +\infty.
\end{aligned}$$

And from (4.37), we have

$$\left|\begin{array}{l} L^*\hat{w} = H + \alpha_1 \hat{h}^1 1_{\mathcal{O}_{1,d}} + \alpha_2 \hat{h}^2 1_{\mathcal{O}_{2,d}} \text{ in } Q, \\ L\hat{h}^i = H_i - \dfrac{1}{\mu_i}\hat{w} 1_{\mathcal{O}_i} \text{ in } Q, \\ \hat{w}(0,t) = \hat{w}(L,t) = 0, \quad \hat{h}^i(0,t) = \hat{h}^i(L,t) = 0, \quad \text{in} \quad (0,T), \\ \hat{w}(T) = 0, \quad \hat{h}^i(0) = 0, \quad \text{in} \quad I, \end{array}\right.$$

where $\hat{w} := \rho_3\rho_1^{-2}\hat{u}$, $\hat{h}^i := \rho_3\rho_1^{-2}\hat{z}^i$, $H := \rho_3\rho_1^{-2}(L^*\hat{u} - \alpha_1\hat{z}^1 1_{\mathcal{O}_{1,d}} - \alpha_2\hat{z}^2 1_{\mathcal{O}_{2,d}}) + (\rho_3\rho_1^{-2})_t\hat{u}$ and $H_i := \rho_3\rho_1^{-2}(L\hat{z}^i + \frac{1}{\mu_i}\hat{u}1_{\mathcal{O}_i}) + (\rho_3\rho_1^{-2})_t\hat{z}^i$.

Using (4.31), we get

$$\begin{aligned} \iint_Q |H|^2 dxdt &\leqslant C\left(\iint_Q \rho_0^{-2}|L^*\hat{u} - \alpha_1\hat{z}^1 1_{\mathcal{O}_{1,d}} - \alpha_2\hat{z}^2 1_{\mathcal{O}_{2,d}}|^2 dxdt + \iint_Q \rho^{-2}|\hat{u}|^2 dxdt\right) \\ &\leqslant Cb((\hat{u},\hat{z}^1,\hat{z}^2),(\hat{u},\hat{z}^1,\hat{z}^2)) \end{aligned}$$

and

$$\begin{aligned} \iint_Q |H_i|^2 dxdt &\leqslant C\left(\iint_Q \rho_0^{-2}|L\hat{z}^i + \frac{1}{\mu_i}\hat{u}1_{\mathcal{O}_i}|^2 dxdt + \iint_Q \rho^{-2}|\hat{z}^i|^2 dxdt\right) \\ &\leqslant Cb((\hat{u},\hat{z}^1,\hat{z}^2),(\hat{u},\hat{z}^1,\hat{z}^2)) \end{aligned}$$

then

$$\rho_3 f \in L^2(0,T;H_0^1(I)\cap H^2(I)), (\rho_3 f)_t \in L^2(Q). \tag{4.39}$$

Furthermore, we have

$$\begin{aligned} \|\rho_3 f\|^2_{L^2(0,T;H_0^1(I))} + \|(\rho_3)_t f\|^2_{L^2(Q)} \leqslant & C\ b((\hat{u},\hat{z}^1,\hat{z}^2),(\hat{u},\hat{z}^1,\hat{z}^2)) \\ := & C\left(\iint_Q \rho_0^2|\hat{y}|^2 dxdt + \sum_{i=1}^2 \iint_Q \rho_0^2|\hat{p}^i|^2 dxdt\right. \\ & \left. + \iint_{\mathcal{O}\times(0,T)} \rho_1^2|\hat{f}|^2 dxdt\right) \end{aligned} \tag{4.40}$$

Note that, from (4.39) one has $f_x \in C([0,T-\delta],L^2(I))$. □

Additional Estimates

From (4.31), we have

$$\begin{aligned} &\rho_i \leqslant C\rho_{i-1} \quad \forall i \in \{\ 1,2,3,4,5\}, \\ &\rho_0 \leqslant C\rho \leqslant C\rho_5^2 \\ &|\rho_i\rho_{i,t}| \leqslant C|\rho_{i-1}|^2, \quad \forall i \in \{\ 2,3,4,5\}\ . \end{aligned} \tag{4.41}$$

Proposition 4.2.6. *Let the hypotheses in Proposition* 4.2.5 *be satisfied and let f and (y, p^1, p^2) satisfy* (4.33)*. Then one has*

$$\begin{aligned}
&\underset{[0,T]}{sup}(\rho_2^2(t)\|y(t)\|^2) + \underset{[0,T]}{sup}(\rho_2^2(t)\|p^i(t)\|^2) + \iint_Q \rho_2^2(|y_x|^2 + |p_x^i|^2)dxdt \\
&\quad + \underset{[0,T]}{sup}(\rho_3^2(t)\|y_x(t)\|^2) + \underset{[0,T]}{sup}(\rho_3^2(t)\|p_x^i(t)\|^2) + \iint_Q \rho_3^2(|y_t|^2 + |y_{xx}|^2 + |p_t^i|^2 + |p_{xx}^i|^2)dxdt \\
\leqslant& C\left(\|y_0\|^2 + \iint_Q \rho^2|G|^2dxdt + \sum_{i=1}^{2}\iint_Q \rho^2|G_i|^2dxdt\right. \\
&\quad \left. + \iint_{\mathcal{O}\times(0,T)} \rho_1^2|f|^2dxdt + \sum_{i=1}^{2}\iint_Q \rho_0^2|p^i|^2dxdt + \iint_Q \rho_0^2|y|^2dxdt\right).
\end{aligned} \tag{4.42}$$

Proof. Multiplying by $\rho_2^2 y$ the equation $(4.16)_1$ and integrating in I, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\rho_2^2\|y(t)\|^2) + \frac{a(0)}{2}\int_I \rho_2^2|y_x|^2dx \\
&\quad \leqslant C\left(\int_I \rho_2^2|G|^2dx + \int_{\mathcal{O}} \rho_2^2|f|^2dx + \int_I \rho_2^2|y|^2dx + \sum_{i=1}^{2}\int_I \rho_2^2|p^i|^2dx\right) + \int_I \rho_{2,t}\rho_2|y|^2dx \\
&\quad \leqslant C\left(\int_I \rho^2|G|^2dx + \int_{\mathcal{O}} \rho_1^2|f|^2dx + \int_I \rho_0^2|y|^2dx + \sum_{i=1}^{2}\int_I \rho_0^2|p^i|^2dx\right).
\end{aligned}$$

Integrating from 0 to t, we have

$$\begin{aligned}
&\sup_{[0,T]}(\rho_2^2\|y(t)\|^2) + \iint_Q \rho_2^2|y_x|^2dxdt \\
&\qquad \leqslant C\left(\|y_0\|^2 + \iint_Q \rho^2|G|^2dxdt + \iint_{\mathcal{O}\times(0,T)} \rho_1^2|f|^2dxdt\right. \\
&\qquad\quad \left. + \iint_Q \rho_0^2|y|^2dxdt + \sum_{i=1}^{2}\iint_Q \rho_0^2|p^i|^2dxdt\right)
\end{aligned} \tag{4.43}$$

Analogously, multiplying $\rho_2^2 p^i$ to the equation $(4.16)_2$ and integrating in Q, we have

$$\begin{aligned}
&\sup_{[0,T]}(\rho_2^2\|p^i(t)\|^2) + \iint_Q \rho_2^2|p_x^i|^2dxdt \\
&\qquad \leqslant C\left(\iint_Q \rho^2|G_i|^2dxdt + \iint_Q \rho_0^2|y|^2dxdt + \iint_Q \rho_0^2|p^i|^2dxdt\right).
\end{aligned} \tag{4.44}$$

Now, multiplying by $\rho_3^2 y_t$ the equation $(4.16)_1$ and integrating in I, we have

$$\begin{aligned}
&\frac{a(0)}{2}\frac{d}{dt}(\rho_3^2\|y_x(t)\|^2) + \int_I \rho_3^2|y_t|^2dx \leqslant a(0)\int_I \rho_{3,t}\rho_3|y_x|^2dx + \epsilon\int_I \rho_3^2|y_t|^2dx \\
&\quad + C_\epsilon\left(\int_I \rho_3^2|G|^2dx + \int_{\mathcal{O}} \rho_3^2|f|^2dx + \int_I \rho_3^2|y|^2dx + \int_I \rho_3^2|y_x|^2dx + \sum_{i=1}^{2}\int_I \rho_3^2|p^i|^2dx\right).
\end{aligned}$$

This is

$$\frac{d}{dt}(\rho_3^2\|y_x(t)\|^2) + \int_I \rho_3^2|y_t|^2dx$$
$$\leqslant C\left(\int_I \rho^2|G|^2dx + \int_{\mathcal{O}}\rho_1^2|f|^2dx + \int_I \rho_0^2|y|^2dx + \sum_{i=1}^{2}\int_I \rho_0^2|p^i|^2dx + \int_I \rho_2^2|y_x|^2dx\right).$$

Integrating from 0 to t and using (4.43), we have

$$\sup_{[0,T]}(\rho_3^2\|y_x(t)\|^2) + \iint_Q \rho_3^2|y_t|^2dxdt$$
$$\leqslant C\left(\|y_0\|^2_{H_0^1(I)} + \iint_Q \rho^2|G|^2dxdt + \iint_{\mathcal{O}\times(0,T)}\rho_1^2|f|^2dxdt\right.$$
$$\left. + \iint_Q \rho_0^2|y|^2dxdt + \sum_{i=1}^{2}\iint_Q \rho_0^2|p^i|^2dxdt\right). \tag{4.45}$$

Now, multiplying by $-\rho_3^2 y_{xx}$ the equation $(4.16)_1$ and integrating in I, we have

$$a(0)\int_I \rho_3^2|y_{xx}|^2dx \leqslant \epsilon\int_I \rho_3^2|y_{xx}|^2dx + C_\epsilon\left(\int_I \rho_3^2|G|^2dx + \int_{\mathcal{O}}\rho_3^2|f|^2dx + \int_I \rho_3^2|y_t|^2dx\right.$$
$$\left. + \int_I \rho_3^2|y|^2dx + \int_I \rho_3^2|y_x|^2dx + \sum_{i=1}^{2}\int_I \rho_3^2|p^i|^2dx\right).$$

then

$$\int_I \rho_3^2|y_{xx}|^2dx \leqslant +C\left(\int_I \rho^2|G|^2dx + \int_{\mathcal{O}}\rho_1^2|f|^2dx + \int_I \rho_3^2|y_t|^2dx\right.$$
$$\left. + \int_I \rho_0^2|y|^2dx + \int_I \rho_2^2|y_x|^2dx + \sum_{i=1}^{2}\int_I \rho_0^2|p^i|^2dx\right).$$

Integrating from 0 to t and using (4.43) and (4.45), we get

$$\iint_Q \rho_3^2|y_{xx}|^2dxdt \leqslant C\left(\|y_0\|^2_{H_0^1(I)} + \iint_Q \rho^2|G|^2dxdt + \iint_{\mathcal{O}\times(0,T)}\rho_1^2|f|^2dxdt\right.$$
$$\left. + \iint_Q \rho_0^2|y|^2dxdt + \sum_{i=1}^{2}\iint_Q \rho_0^2|p^i|^2dxdt\right) \tag{4.46}$$

Analogously, multiplying $\rho_3^2 p_t^i$ to the equation $(4.16)_2$ and integrating in Q, we have

$$\sup_{[0,T]}(\rho_3^2\|p_x^i(t)\|^2) + \iint_Q \rho_3^2|p_t^i|^2dxdt$$
$$\leqslant C\left(\iint_Q \rho^2|G_i|^2dxdt + \iint_Q \rho_0^2|y|^2dxdt + \iint_Q \rho_0^2|p^i|^2dxdt\right) \tag{4.47}$$

and also multiplying $-\rho_3^2 p^i_{xx}$ to the equation $(4.16)_2$ and integrating in Q, we have

$$\iint_Q \rho_3^2|p^i_{xx}|^2 dxdt \leqslant C\left(\iint_Q \rho^2|G_i|^2dxdt + \iint_Q \rho_0^2|y|^2dxdt + \iint_Q \rho_0^2|p^i|^2dxdt\right) \tag{4.48}$$

From (4.43) – (4.48), we have (4.42). □

Proposition 4.2.7. *Let the hypotheses in Proposition* 4.2.5 *be satisfied and let* f *and* (y, p^1, p^2) *satisfy* (4.33)*. Then one has*

$$\begin{aligned}
&\sup_{[0,T]}(\rho_4^2(t)\|y_t(t)\|^2) + \iint_Q \rho_4^2|y_{xt}|^2dxdt \\
&+\sup_{[0,T]}(\rho_5^2(t)\|y_{xt}(t)\|^2) + \iint_Q \rho_5^2(|y_{tt}|^2 + |y_{xxt}|^2)dxdt + \sup_{[0,T]}(\rho_5^2(t)\|y_{xx}(t)\|^2) \\
&\leqslant C\left(\|y_0\|^2_{H^3_0(I)} + \|G(0)\|^2_{H^1_0(I)} + \iint_Q \rho^2|G|^2dxdt + \iint_Q \rho_3^2|G_t|^2dxdt\right. \\
&\quad \left. + \sum_{i=1}^{2}\iint_Q \rho^2|G_i|^2dxdt + \iint_{\mathcal{O}\times(0,T)} \rho_1^2|f|^2dxdt + \sum_{i=1}^{2}\iint_Q \rho_0^2|p^i|^2dxdt + \iint_Q \rho_0^2|y|^2dxdt\right).
\end{aligned} \tag{4.49}$$

Proof. We know that

$$y_{tt} - a(0)y_{xxt} + D_1F(0,0)y_t + D_2F(0,0)y_{xt} = f_t1_{\mathcal{O}} - \frac{1}{\mu_1}p^1_t1_{\mathcal{O}_1} - \frac{1}{\mu_2}p^2_t1_{\mathcal{O}_2} + G_t. \tag{4.50}$$

From (4.50) multiplying by $\rho_4^2 y_t$ and integrating in I, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\rho_4^2(t)\|y_t(t)\|^2dx\right) + \frac{a(0)}{2}\int_I \rho_4^2|y_{xt}|^2dx \leqslant \int_I \rho_4\rho_{4,t}|y_t|^2dx \\
&\quad + C\left(\int_I \rho_4^2|G_t|^2dx + \int_{\mathcal{O}} \rho_4^2|f_t|^2dx + \int_I \rho_4^2|y_t|^2dx + \sum_{i=1}^{2}\int_I \rho_4^2|p^i_t|^2dx\right).
\end{aligned}$$

Integrating from 0 to t and using (4.40) and (4.42), we have

$$\begin{aligned}
\sup_{[0,T]}(\rho_4^2(t)\|y_t(t)\|^2) + \iint_Q \rho_4^2|y_{xt}|^2dxdt \leqslant C&\left(\|y_t(0)\|^2 + \iint_Q \rho_3^2|G_t|^2dxdt\right. \\
&\left. + \iint_{\mathcal{O}\times(0,T)} \rho_1^2|f|^2dxdt + \iint_Q \rho_0^2|y|^2dxdt + \sum_{i=1}^{2}\iint_Q \rho_0^2|p^i|^2dxdt\right).
\end{aligned} \tag{4.51}$$

We get easily that

$$\|y_t(0)\| \leqslant C(\|y(0)\|_{H^2(I)} + \|f(0)\|_{L^2(\mathcal{O})} + \|p^1(0)\| + \|p^2(0)\| + \|G(0)\|).$$

Since $\rho_3 f 1_{\mathcal{O}}, \rho_3 G \in H^1(0,T;L^2(I))$, in (4.51) one has

$$\sup_{[0,T]}(\rho_4^2(t)\|y_t(t)\|^2) + \iint_Q \rho_4^2|y_{xt}|^2 dxdt \leqslant C\Bigg(\|y_0\|^2_{H^2(I)} + \iint_Q \rho^2|G|^2 dxdt + \iint_Q \rho_3^2|G_t|^2 dxdt + \sum_{i=1}^2 \iint_Q \rho^2|G_i|^2 dxdt + \iint_{\mathcal{O}\times(0,T)} \rho_1^2|f|^2 dxdt + \iint_Q \rho_0^2|y|^2 dxdt + \sum_{i=1}^2 \iint_Q \rho_0^2|p^i|^2 dxdt\Bigg). \tag{4.52}$$

From (4.50) multiplying by $\rho_5^2 y_{tt}$ and integrating in I, we have

$$\int_I \rho_5^2|y_{tt}|^2 dx + \frac{a(0)}{2}\frac{d}{dt}\left(\int_I \rho_5^2|y_{xt}|^2 dx\right) \leqslant a(0)\int_I \rho_5\rho_{5,t}|y_{xt}|^2 dx + \epsilon\int_I \rho_5^2|y_{tt}|^2 dx + C_\epsilon\left(\int_I \rho_5^2|G_t|^2 dx + \int_{\mathcal{O}} \rho_5^2|f_t|^2 dx + \int_I \rho_5^2|y_t|^2 dx + \int_I \rho_5^2|y_{xt}|^2 dx + \sum_{i=1}^2 \int_I \rho_5^2|p_t^i|^2 dx\right)$$

then

$$\int_I \rho_5^2|y_{tt}|^2 dx + \frac{d}{dt}\int_I \rho_5^2|y_{xt}|^2 dx \leqslant C\Bigg(\int_I \rho_3^2|G_t|^2 dx + \int_{\mathcal{O}} \rho_3^2|f_t|^2 dx + \int_I \rho_3^2|y_t|^2 dx + \int_I \rho_4^2|y_{xt}|^2 dx + \sum_{i=1}^2 \int_I \rho_3^2|p_t^i|^2 dx\Bigg).$$

Integrating from 0 to t, using (4.40), (4.45), (4.47) and (4.52) we deduce

$$\iint_Q \rho_5^2|y_{tt}|^2 dxdt + \sup_{[0,T]}\left(\rho_5^2(t)\|y_t(t)\|^2\right) \leqslant C\Bigg(\|y_{xt}(0)\|^2 + \iint_Q \rho_3^2|G_t|^2 dxdt + \iint_{\mathcal{O}\times(0,T)} \rho_1^2|f|^2 dxdt + \iint_Q \rho_0^2|y|^2 dxdt + \sum_{i=1}^2 \iint_Q \rho_0^2|p^i|^2 dxdt\Bigg).$$

We see easily that

$$\|y_{x,t}(0)\| \leqslant C(\|y_0\|_{H^3(I)} + \|f_x(0)\|_{L^2(\mathcal{O})} + \|p_x^1(0)\| + \|p_x^2(0)\| + \|G_x(0)\|).$$

Then using (4.47) and (4.40), we deduce

$$\iint_Q \rho_5^2|y_{tt}|^2 dxdt + \sup_{[0,T]}\left(\rho_5^2(t)\|y_t(t)\|^2\right) \leqslant C\Bigg(\|y_0\|^2_{H^3(I)} + \|G(0)\|^2_{H_0^1(I)} + \iint_Q \rho_3^2|G_t|^2 dxdt + \sum_{i=1}^2 \iint_Q \rho^2|G_i|^2 dxdt + \iint_Q \rho_1^2|f|^2 dxdt + \iint_Q \rho_0^2|y|^2 dxdt + \sum_{i=1}^2 \iint_Q \rho_0^2|p^i|^2 dxdt\Bigg). \tag{4.53}$$

Analogously from (4.50) multiplying by $\rho_5^2 y_{xxt}$ and integrating in Q, we have

$$\sup_{[0,T]} \left(\rho_5^2(t)\|y_{xt}(t)\|^2\right) + \iint_Q \rho_5^2|y_{xxt}|^2 dxdt \leqslant C \Big(\|y_0\|^2_{H^3(I)} + \|G(0)\|^2_{H_0^1(I)} + \iint_Q \rho_3^2|G_t|^2 dxdt + \sum_{i=1}^2 \iint_Q \rho^2|G_i|^2 dxdt + \iint_Q \rho_1^2|f|^2 dxdt + \iint_Q \rho_0^2|y|^2 dxdt + \sum_{i=1}^2 \iint_Q \rho_0^2|p^i|^2 dxdt \Big). \tag{4.54}$$

Also, from $(4.16)_1$ multiplying by $-\rho_5^2 y_{xxt}$ and integrating in I, we have

$$\int_I \rho_5^2|y_{xt}|^2 dx + \frac{a(0)}{2}\frac{d}{dt}\left(\rho_5^2(t)\|y_{xx}(t)\|^2\right) \leqslant a(0)\int_I \rho_5\rho_{5,t}|y_{xx}|^2 dx + C\Big(\int_{\mathcal{O}} \rho_5^2|f|^2 dx + \int_I \rho_5^2|y|^2 dx + \int_I \rho_5^2|y_x|^2 dx + \int_I \rho_5^2|y_{xxt}|^2 dx + \sum_{i=1}^2 \int_I \rho_5^2|p^i|^2 dx + \int_I \rho_5^2|G|^2 dx\Big)$$

whence

$$\iint_Q \rho_5^2|y_{xt}|^2 dxdt + \sup_{[0,T]} \left(\rho_5^2(t)\|y_{xx}(t)\|^2\right) \leqslant C \Big(\|y_0\|^2_{H^3(I)} + \|G(0)\|^2_{H_0^1(I)} + \iint_Q \rho_3^2|G_t|^2 dxdt + \sum_{i=1}^2 \iint_Q \rho^2|G_i|^2 dxdt + \iint_Q \rho_1^2|f|^2 dxdt + \iint_Q \rho_0^2|y|^2 dxdt + \sum_{i=1}^2 \iint_Q \rho_0^2|p^i|^2 dxdt \Big). \tag{4.55}$$

Gathering (4.52) – (4.55) , we have (4.49). □

4.2.3 Null Controllability for (4.15)

In this section we will prove the null controllability for the optimal system using Right Inverse Function theorem for Banach spaces.

Let us introduce the space

$$\begin{aligned} Y := \{ &(y, p^1, p^2, f); \rho_0 y, \rho_0 p^i \in L^2(Q); \rho_1 f \in L^2(\mathcal{O}\times(0,T)); \\ &\rho(y_t - a(0)y_{xx} + D_1F(0,0)y + D_2F(0,0)y_x - f1_{\mathcal{O}} + \frac{1}{\mu_1}p^1 1_{\mathcal{O}_1} + \frac{1}{\mu_2}p^2 1_{\mathcal{O}_2}), \\ &\rho_3(y_{tt} - a(0)y_{xxt} + D_1F(0,0)y_t + D_2F(0,0)y_{xt} - f_t 1_{\mathcal{O}}) \in L^2(Q), \\ &\rho(-p^i_t - a(0)p^i_{xx} + D_1F(0,0)p^i - D_2F(0,0)p^i_x - \alpha_i y 1_{\mathcal{O}_{i,d}}) \in L^2(Q); \\ &y(0) \in H^3(I)\cap H_0^1(I) \,\} \end{aligned}$$

with norm

$$\begin{aligned}
\|(y,p^1,p^2,f)\|_Y^2 &:= \|\rho_0 y\|^2_{L^2(Q)} + \sum_{i=1}^2 \|\rho_0 p^i\|^2_{L^2(Q)} + \|\rho_1 f\|^2_{L^2(\mathcal{O}\times(0,T))} \\
&+ \|\rho(y_t - a(0)y_{xx} + D_1F(0,0)y + D_2F(0,0)y_x - f1_{\mathcal{O}} + \frac{1}{\mu_1}p^1 1_{\mathcal{O}_1} + \frac{1}{\mu_2}p^2 1_{\mathcal{O}_2})\|^2_{L^2(Q)} \\
&+ \|\rho_3(y_{tt} - a(0)y_{xxt} + D_1F(0,0)y_t + D_2F(0,0)y_{xt} - f_t 1_{\mathcal{O}})\|^2_{L^2(Q)} + \|y(0)\|^2_{H^3(I)} \\
&+ \sum_{i=1}^2 \|\rho(-p^i_t - a(0)p_{xx} + D_1F(0,0)p^i - D_2F(0,0)p^i_x + \alpha_i y 1_{\mathcal{O}_{i,d}})\|^2_{L^2(Q)}.
\end{aligned}$$

It is clear that Y is a Hilbert space equipped with the norm $\|\cdot\|_Y$.

Let $L^2(\rho^2;Q)$ be the Hilbert space formed by the measurable functions $w = w(x,t)$ such that $\rho w \in L^2(Q)$, i.e.

$$\|w\|^2_{L^2(\rho^2;Q)} := \iint_Q \rho^2 |w|^2 dxdt < +\infty$$

and $F := \{g \in L^2(Q); \rho g, \rho_3 g_t \in L^2(Q), g(0) \in H^1_0(I)\}$.

Let us introduce the Hilbert space

$$Z := F \times (L^2(\rho^2;Q))^2 \times (H^3(I) \cap H^1_0(I))$$

with norm

$$\begin{aligned}
\|(G,G_1,G_2,y_0)\|^2_Z &:= \|\rho G\|^2_{L^2(Q)} + \|\rho_3 G_t\|^2_{L^2(Q)} + \|G(0)\|^2_{H^1_0(I)} + \|\rho G_1\|^2_{L^2(Q)} \\
&+ \|\rho G_2\|^2_{L^2(Q)} + \|y_0\|^2_{H^3(I)}.
\end{aligned}$$

Observation 4.2.8. *Notice that, if $(y,p^1,p^2,f) \in Y$, in view of Propositions 4.2.6 and 4.2.7, one has*

$$\begin{aligned}
&\sup_{[0,T]}(\rho_3^2(t)\|y_x(t)\|^2) + \sup_{[0,T]}(\rho_3^2(t)\|p_x(t)\|^2) + \iint_Q \rho_3^2(|y_t|^2 + |y_{xx}|^2 + |p_t|^2 + |p_{xx}|^2)dxdt \\
&+ \sup_{[0,T]}(\rho_4^2(t)\|y_t(t)\|^2) + \iint_Q \rho_4^2 |y_{xt}|^2 dxdt + \sup_{[0,T]}(\rho_5^2(t)\|y_{xx}(t)\|^2) + \iint_Q \rho_5^2 |y_{xxt}|^2 dxdt \\
&\leqslant C\|(y,p^1,p^2,f)\|^2_Y.
\end{aligned}$$

Let us define the mapping $\mathcal{A} : Y \to Z$, given by

$$\mathcal{A}(y,p^1,p^2,f) := (\mathcal{A}_1(y,p^1,p^2,f), \mathcal{A}_2(y,p^1,p^2,f), \mathcal{A}_3(y,p^1,p^2,f), \mathcal{A}_4(y,p^1,p^2,f)) \tag{4.56}$$

where

$$
\begin{aligned}
\mathcal{A}_1(y,p^1,p^2,f) :=& y_t - (a(y_x)y_x)_x + F(y,y_x) - f1_{\mathcal{O}} + \frac{1}{\mu_1}p^1 1_{\mathcal{O}_1} + \frac{1}{\mu_2}p^2 1_{\mathcal{O}_2},\\
\mathcal{A}_2(y,p^1,p^2,f) :=& -p^1_t - ((a'(y_x)y_x + a(y_x))p^1_x)_x\\
&+ D_1F(y,y_x)p^1 - (D_2F(y,y_x)p^1)_x - \alpha_1 y 1_{\mathcal{O}_{1,d}},\\
\mathcal{A}_3(y,p^1,p^2,f) :=& -p^2_t - ((a'(y_x)y_x + a(y_x))p^2_x)_x\\
&+ D_1F(y,y_x)p^2 - (D_2F(y,y_x)p^2)_x - \alpha_2 y 1_{\mathcal{O}_{2,d}},\\
\mathcal{A}_4(y,p^1,p^2,f) :=& y(0).
\end{aligned}
$$

We will use the following lemmas to conclude the desired result.

Lemma 4.2.9. *Let $\mathcal{A} : Y \to Z$ be the mapping defined by (4.56). Then, $\mathcal{A}$ is well defined and continuous.*

Proof. For every $(y,p^1,p^2,f) \in Y$ one has

$$
\begin{aligned}
&\|\rho\mathcal{A}_1(y,p^1,p^2,f)\|^2_{L^2(Q)}\\
=&\iint_Q \rho^2 |y_t - (a(y_x)y_x)_x + F(y,y_x) - f1_{\mathcal{O}} + \frac{1}{\mu_1}p^1 1_{\mathcal{O}_1} + \frac{1}{\mu_2}p^2 1_{\mathcal{O}_2}|^2 dxdt\\
\leqslant& C\Bigg(\iint_Q \rho_0^2 |y_t - a(0)y_{xx} + D_1F(0,0)y + D_2F(0,0)y_x - f1_{\mathcal{O}}\\
&+ \frac{1}{\mu_1}p^1 1_{\mathcal{O}_1} + \frac{1}{\mu_2}p^2 1_{\mathcal{O}_2}|^2 dxdt + \iint_Q \rho_0^2 |a(y_x) - a(0)|^2 |y_{xx}|^2 dxdt\\
&+ \iint_Q \rho^2 |a'(y_x)|^2 |y_x|^2 |y_{xx}|^2 dxdt + \iint_Q \rho^2 |F(y,y_x) - D_1F(0,0)y - D_2F(0,0)y_x|^2 dxdt\Bigg)\\
=& I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

We know that

$$
I_1 \leqslant \|(y,p^1,p^2,f)\|^2_Y < +\infty.
$$

Also

$$
\begin{aligned}
I_2 + I_3 &\leqslant C \iint_Q \rho^2 |y_x|^2 |y_{xx}|^2 dxdt\\
&\leqslant C \int_0^T \rho_5^2 \rho_3^2 \|y_{xx}(t)\|^2 \|y_{xx}(t)\|^2 dt\\
&\leqslant C \left(\sup_{[0,T]} \ \rho_5^2(t) \|y_{xx}(t)\|^2 \right) \iint_Q \rho_3^2 |y_{xx}|^2 dxdt\\
&\leqslant C \|(y,p^1,p^2,f)\|^4_Y < +\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 \leqslant & C \iint_Q \rho^2 (|\nabla F(\tilde{\theta}(y,y_x)) - \nabla F(0,0)|^2 (|y|^2 + |y_x|^2) dxdt \\
\leqslant & \; C \iint_Q \rho^2 \tilde{\theta}^2 (|y|^2 + |y_x|^2)(|y|^2 + |y_x|^2) dxdt \\
\leqslant & \; C \int_0^T \rho_5^2 \rho_3^2 (\|y_x(t)\|^2 + \|y_{xx}(t)\|^2)(\|y(t)\|^2 + \|y_x(t)\|^2) \\
\leqslant & \; C \left\{ \left(\sup_{[0,T]} \; \rho_5^2(t) \|y_{xx}(t)\|^2 \right) + \left(\sup_{[0,T]} \; \rho_3^2(t) \|y_x(t)\|^2 \right) \right\} \cdot \\
& \left(\iint_Q \rho_0^2 |y|^2 dxdt + \iint_Q \rho_2^2 |y_x|^2 dxdt \right) \\
\leqslant & \; C \|(y, p^1, p^2, f)\|_Y^4 < +\infty,
\end{aligned}
$$

where $\tilde{\theta} := \tilde{\theta}(x,t) \in (0,1)$.

Now

$$
\begin{aligned}
& \|\rho_3 \mathcal{A}_{1,t}(y, p^1, p^2, f)\|_{L^2(Q)} \\
& = \iint_Q \rho_3^2 |(y_t - (a(y_x) y_x)_x + F(y, y_x) - f 1_{\mathcal{O}} + \frac{1}{\mu_1} p^1 1_{\mathcal{O}_1} + \frac{1}{\mu_2} p^2 1_{\mathcal{O}_2})_t|^2 dxdt \\
& = \iint_Q \rho_3^2 |y_{tt} - a(y_x) y_{xxt} - a'(y_x) y_x y_{xxt} - 2a(y_x) y_{xt} y_{xx} - a''(y_x) y_{xt} y_{xx} y_x \\
& \quad + \nabla F(y, y_x)(y, y_x)_t - f_t 1_{\mathcal{O}} + \frac{1}{\mu_1} p_t^1 1_{\mathcal{O}_1} + \frac{1}{\mu_2} p_t^2 1_{\mathcal{O}_2}|^2 dxdt \\
& \leqslant C \left(\iint_Q \rho_3^2 |y_{tt} - a(0) y_{xxt} + \nabla F(0,0)(y, y_x)_t - f_t 1_{\mathcal{O}}|^2 dxdt \right. \\
& \quad + \iint_Q \rho_3^2 |a(y_x) - a(0)|^2 |y_{xxt}|^2 dxdt + \iint_Q \rho_3^2 |a'(y_x)|^2 |y_{xxt}|^2 |y_x|^2 dxdt \\
& \quad + \iint_Q \rho_3^2 |a'(y_x)|^2 |y_{xt}|^2 |y_{xx}|^2 dxdt + \iint_Q \rho_3^2 |a''(y_x)|^2 |y_{xt}|^2 |y_{xx}|^2 |y_x|^2 dxdt \\
& \quad \left. + \iint_Q \rho_3^2 (|y|^2 + |y_x|^2)(|y_t|^2 + |y_{xt}|^2) dxdt + \sum_{i=1}^2 \iint_Q \rho_3^2 |p_t^i|^2 dxdt \right) \\
& \leqslant C \left(\iint_Q \rho_3^2 |y_{tt} - a(0) y_{xxt} + \nabla F(0,0)(y, y_x)_t - f_t 1_{\mathcal{O}}|^2 dxdt \right. \\
& \quad + \iint_Q \rho_3^2 |y_x|^2 |y_{xxt}|^2 dxdt + \iint_Q \rho_3^2 |y_{xxt}|^2 |y_x|^2 dxdt \\
& \quad + \iint_Q \rho_3^2 |y_{xt}|^2 |y_{xx}|^2 dxdt + \iint_Q \rho_3^2 |y_{xt}|^2 |y_{xx}|^2 |y_x|^2 dxdt \\
& \quad \left. + \iint_Q \rho_3^2 (|y|^2 + |y_x|^2)(|y_t|^2 + |y_{xt}|^2) dxdt + \sum_{i=1}^2 \iint_Q \rho_3^2 |p_t^i|^2 dxdt \right) \\
& \leqslant C \left(\|(y, p^1, p^2, f)\|_Y^2 + \|(y, p^1, p^2, f)\|_Y^4 + \|(y, p^1, p^2, f)\|_Y^6 \right) < +\infty.
\end{aligned}
$$

Analogously, we have

$$
\begin{aligned}
\|\mathcal{A}_1(y,p^1,p^2,f)(0)\|_{H_0^1(I)} &= \int_I |(y_{xt}(0)-(a(y_x(0))y_x(0))_{xx} - f_x(0)1_{\mathcal{O}} + \frac{1}{\mu}p_x(0)1_\omega|^2 dx \\
&\leqslant C(\|(y,p^1,p^2,f)\|_Y^2 + \|(y,p^1,p^2,f)\|_Y^4) < +\infty.
\end{aligned}
$$

And finally

$$
\begin{aligned}
&\|\rho\mathcal{A}_{i+1}(y,p^1,p^2,f)\|^2_{L^2(Q)} \\
&= \iint_Q \rho^2 | -p_t^i - ((a'(y_x)y_x + a(y_x))p_x^i)_x + D_1F(y,y_x)p^i - (D_2F(y,y_x)p^i)_x \\
&\quad - \alpha_i y 1_{\mathcal{O}_{i,d}}|^2 dxdt \\
&\leqslant C\left(\iint_Q \rho^2 | -p_t^i - a(0)p_{xx}^i + D_1F(0,0)p^i - D_2F(0,0)p_x^i - \alpha_i y 1_{\mathcal{O}_{i,d}}|^2 dxdt\right. \\
&\quad + \iint_Q \rho^2 |a(y_x)-a(0)|^2|p_{xx}^i|^2 dxdt + \iint_Q \rho^2|a'(y_x)|^2|p_x^i|^2|y_{xx}|^2 dxdt \\
&\quad + \iint_Q \rho^2|a''(y_x)|^2|y_{xx}|^2|y_x|^2|p_x|^2 dxdt + \iint_Q \rho_3^2|D_1(F(y,y_x)-F(0,0))|^2|p^i|^2 dxdt \\
&\quad + \iint_Q \rho_3^2|D_2(F(y,y_x)-F(0,0))|^2|p_x^i|^2 dxdt + \iint_Q \rho_3^2|D_{12}^2(F(y,y_x)|^2|y_x|^2|p^i|^2 dxdt \\
&\quad \left. + \iint_Q \rho_3^2|D_2^2F(y,y_x)|^2|y_{xx}|^2|p^i|^2 dxdt\right) \\
&\leqslant C\left(\iint_Q \rho^2 | -p_t^i - a(0)p_{xx}^i + D_1F(0,0)p^i - D_2F(0,0)p_x^i - \alpha_i y 1_{\mathcal{O}_{i,d}}|^2 dxdt\right. \\
&\quad + \iint_Q \rho_3^2\rho_5^2|y_x|^2|p_{xx}^i|^2 dxdt + \iint_Q \rho_3^2\rho_5^2|p_x^i|^2|y_{xx}|^2 dxdt \\
&\quad + \iint_Q \rho_3^4\rho_5^2|y_{xx}|^2|y_x|^2|p_x|^2 dxdt + \iint_Q \rho_3^2\rho_5^2(|y|^2+|y_x|^2)|p^i|^2 dxdt \\
&\quad + \iint_Q \rho_3^2\rho_5^2(|y|^2+|y_x|^2)|p_x^i|^2 dxdt + \iint_Q \rho_3^2\rho_5^2|y_x|^2|p^i|^2 dxdt \\
&\quad \left. + \iint_Q \rho_3^2\rho_5^2|y_{xx}|^2|p^i|^2 dxdt\right) \\
&\leqslant C\left(\|(y,p^1,p^2,f)\|_Y^2 + \|(y,p^1,p^2,f)\|_Y^4 + \|(y,p^1,p^2,f)\|_Y^6\right) < +\infty.
\end{aligned}
$$

Consequently, $\mathcal{A}$ takes values em Z.

That the mapping $\mathcal{A}$ is continuous is easy to prove using similar arguments. □

Lemma 4.2.10. *The mapping $\mathcal{A} : Y \to Z$ is continuously differentiable.*

Proof. Let us first prove that $\mathcal{A}$ is $G-$differentiable at any $(y,p^1,p^2,f) \in Y$ and let us compute the $G-$derivative $\mathcal{A}'(y,p^1,p^2,f)$.

Thus, let us fix (y, p^1, p^2, f) in Y and let us take $(y', p^{1'}, p^{2'}, f') \in Y$ and $\lambda > 0$. Let us introduce the linear mapping $D\mathcal{A} : Y \to Z$, with

$$D\mathcal{A}(y, p^1, p^2, f) = D\mathcal{A} = (D\mathcal{A}_1, D\mathcal{A}_2, D\mathcal{A}_3).$$

$$\begin{aligned}
D\mathcal{A}_1(y', p^{1'}, p^{2'}, f') &:= y'_t - (a'(y_x)y'_x y_x)_x - (a(y_x)y'_x)_x \\
&\quad + D_1F(y, y_x)y' + D_2F(y, y_x)y'_x \\
&\quad - f'1_{\mathcal{O}} + \frac{1}{\mu_1}p^{1'}1_{\mathcal{O}_1} + \frac{1}{\mu_2}p^{2'}1_{\mathcal{O}_2}, \\
D\mathcal{A}_{1,t}(y', p^{1'}, p^{2'}, f') &:= y'_{tt} - (a'(y_x)y'_x y_x)_{xt} - (a(y_x)y'_x)_{xt} \\
&\quad + D^2_{11}F(y, y_x)y_t y' + D^2_{12}F(y, y_x)y_{xt}y' \\
&\quad + D^2_{21}F(y, y_x)y_t y'_x + D^2_{22}F(y, y_x)y_{xt}y'_x \\
&\quad + D_1F(y, y_x)y'_t + D_2F(y, y_x)y'_{xt} \\
&\quad - f'_t1_{\mathcal{O}} + \frac{1}{\mu_1}{p^{1'}}_t1_{\mathcal{O}_1} + \frac{1}{\mu_2}{p^{2'}}_t1_{\mathcal{O}_2}, \\
D\mathcal{A}_2(y', p^{1'}, p^{2'}, f') &:= -{p^{1'}}_t - (a'(y_x)y_x {p^{1'}}_x)_x - (a(y_x){p^{1'}}_x)_x - (a''(y_x)y'_x y_x p^1_x)_x \\
&\quad - 2(a'(y_x)y'_x p^1_x)_x + D^2_{11}F(y, y_x)y'p^1 + D^2_{12}F(y, y_x)y'_x p^1 \\
&\quad - (D^2_{21}F(y, y_x)y'p^1)_x - (D^2_{22}F(y, y_x)y'_x p^1)_x \\
&\quad + D_1F(y, y_x)p^{1'} - (D_2F(y, y_x)p^{1'})_x - \alpha_1 y'1_{\mathcal{O}_1}, \\
D\mathcal{A}_3(y', p^{1'}, p^{2'}, f') &:= -{p^{2'}}_t - (a'(y_x)y_x {p^{2'}}_x)_x - (a(y_x){p^{2'}}_x)_x - (a''(y_x)y'_x y_x p^2_x)_x \\
&\quad - 2(a'(y_x)y'_x p^2_x)_x + D^2_{11}F(y, y_x)y'p^2 + D^2_{12}F(y, y_x)y'_x p^2 \\
&\quad - (D^2_{21}F(y, y_x)y'p^2)_x - (D^2_{22}F(y, y_x)y'_x p^2)_x \\
&\quad + D_1F(y, y_x)p^{2'} - (D_2F(y, y_x)p^{2'})_x - \alpha_2 y'1_{\mathcal{O}_2}, \\
D\mathcal{A}_4(y', p^{1'}, p^{2'}, f') &:= y'(0),
\end{aligned}$$

for all $(y', p^{1'}, p^{2'}, f') \in Y$.

From the definition of the spaces Y and Z, it becomes that $D\mathcal{A} \in \mathcal{L}(Y; Z)$.

Furthermore, we have

$$\frac{1}{\lambda}[\mathcal{A}_i((y, p^1, p^2, f) + \lambda(y', p^{1'}, p^{2'}, f')) - \mathcal{A}_i(y, p^1, p^2, f)] \to D\mathcal{A}_i(y', p^{1'}, p^{2'}, f')$$

strongly in $L^2(\rho^2; Q)$ for $i = 1, 2, 3, 4$ as $\lambda \to 0$ and

$$\frac{1}{\lambda}[\mathcal{A}_{1,t}((y, p^1, p^2, f) + \lambda(y', p^{1'}, p^{2'}, f')) - \mathcal{A}_{1,t}(y, p^1, p^2, f)] \to D\mathcal{A}_{1,t}(y', p^{1'}, p^{2'}, f')$$

strongly in $L^2(\rho_3^2;Q)$ as $\lambda \to 0$.

Indeed, we denote

$$a_\lambda := a(y_x + \lambda y'_x), \quad \overline{a} := a(y_x), \quad a'_\lambda := a'(y_x + \lambda y'_x), \quad \overline{a}' := a'(y_x),$$

$$F_\lambda := F(y + \lambda y', y_x + \lambda y'_x), \quad \overline{F} := F(y, y_x),$$

$$F'_{n,i} := D_i F(y^n, y^n_x), \quad F'_{\lambda,i} := D_i F(y + \lambda y', y_x + \lambda y'_x), \quad \overline{F}'_i := D_i F(y, y_x)$$

and we have

$$\begin{aligned}
&\|\frac{1}{\lambda}[\mathcal{A}_1((y,p^1,p^2,f)+\lambda(y',p^{1'},p^{2'},f')) - \mathcal{A}_1(y,p^1,p^2,f)] - D\mathcal{A}_1(y',p^{1'},p^{2'},f')\|^2_{L^2(\rho^2;Q)}\\
\leqslant& C\left(\iint_Q \rho^2 \left|\left(\left[\frac{a_\lambda - \overline{a}}{\lambda} - \overline{a}' y'_x\right] y_x\right)_x\right|^2 dxdt + \iint_Q \rho^2 |(a_\lambda - \overline{a}) y'_x)_x|^2 dxdt\right.\\
&\left.+ \iint_Q \rho^2 \left|\frac{F_\lambda - \overline{F}}{\lambda} - \nabla F(y,y_x)(y',y'_x)\right|^2 dxdt\right)\\
\leqslant& C\left(\iint_Q \rho^2 \left|\frac{a_\lambda - \overline{a}}{\lambda} - \overline{a}' y'_x\right|^2 |y_{xx}|^2 dxdt\right.\\
&+ \iint_Q \rho^2 \left|\frac{a'_\lambda(y_{xx} + \lambda y'_{xx}) - \overline{a}' y_{xx}}{\lambda} - a''(y_x) y_{xx} y'_x - \overline{a}' y'_{xx}\right|^2 |y_x|^2 dxdt\\
&+ \iint_Q \rho^2 |a_\lambda - \overline{a}|^2 |y'_{xx}|^2 dxdt + \iint_Q \rho^2 |a'_\lambda(y_{xx} + \lambda y'_{xx}) - \overline{a}' y_{xx}|^2 |y'_x|^2 dxdt\\
&\left.+ \iint_Q \rho^2 |\nabla F(y + \tilde{\lambda} y', y_x + \tilde{\lambda} y'_x) - \nabla F(y,y_x)|^2 (|y'|^2 + |y'_x|^2) dxdt\right)\\
\leqslant& C\left(\lambda^2 \iint_Q \rho^2 |y'_x|^4 |y_{xx}|^2 dxdt + \iint_Q \rho^2 \left|\frac{a'_\lambda - \overline{a}'}{\lambda} - a''(y_x) y'_x\right|^2 |y_{xx}|^2 |y_x|^2 dxdt\right.\\
&+ \lambda^2 \iint_Q \rho^2 \left|y'_x\right|^2 |y'_{xx}|^2 |y_x|^2 dxdt + \lambda^2 \iint_Q \rho^2 |y'_x|^2 |y'_{xx}|^2 dxdt\\
&\left.+\lambda^2 \iint_Q \rho^2 |y_{xx}|^2 |y'_x|^4 dxdt + \lambda^2 \iint_Q \rho^2 |y'_{xx}|^2 |y'_x|^2 dxdt + \lambda^2 \iint_Q \rho^2 (|y'|^4 + |y'_x|^4)\right),
\end{aligned}$$

where $\tilde{\lambda} := \tilde{\lambda}(x,t) \in (0,\lambda)$.

Using Observation 4.2.8 and Lebesgue's Theorem, we find that

$$\frac{1}{\lambda}[\mathcal{A}_1((y,p^1,p^2,f)+\lambda(y',p^{1'},p^2 1,f')) - \mathcal{A}_1(y,p^1,p^2,f)] \to D\mathcal{A}_1(y',p^{1'},p^{2'},f').$$

Similarly

$$
\begin{aligned}
&\|\frac{1}{\lambda}[\mathcal{A}_{1,t}((y,p^1,p^2,f)+\lambda(y',p^{1'},p^{2'},f'))-\mathcal{A}_{1,t}(y,p^1,p^2,f)]-D\mathcal{A}_{1,t}(y',p^{1'},p^{2'},f')\|^2_{L^2(\rho_3^2;Q)}\\
&\leqslant C\left(\iint_Q \rho_3^2\left|\left(\left[\frac{a_\lambda-\overline{a}}{\lambda}-a'(y_x)y'_x\right]y_x\right)_{xt}\right|^2 dxdt+\iint_Q \rho_3^2|(a_\lambda-\overline{a})y'_x)_{xt}|^2dxdt\right.\\
&+\iint_Q \rho_3^2\left|\frac{F'_{\lambda,1}-\overline{F}'_1}{\lambda}-D^2_{11}F(y,y_x)y'-D^2_{12}F(y,y_x)y'_x\right|^2|y_t|^2dxdt\\
&+\iint_Q \rho_3^2\left|\frac{F'_{\lambda,2}-\overline{F}'_2}{\lambda}-D^2_{21}F(y,y_x)y'-D^2_{22}F(y,y_x)y'_x\right|^2|y_{xt}|^2dxdt\\
&\left.+\iint_Q \rho_3^2|F'_{\lambda,1}-\overline{F}'_1|^2|y'_t|^2dxdt+\iint_Q \rho_3^2|F'_{\lambda,2}-\overline{F}'_2|^2|y'_{xt}|^2dxdt\right)\\
&\leqslant C\left(\iint_Q \rho_3^2\left|\frac{a_\lambda-\overline{a}}{\lambda}-a'(y_x)y'_x\right|^2|y_{xxt}|^2dxdt\right.\\
&+\iint_Q \rho_3^2\left|\left(\frac{a_\lambda-\overline{a}}{\lambda}-a'(y_x)y'_x\right)_t\right|^2|y_{xx}|^2dxdt\\
&+\iint_Q \rho_3^2\left|\left(\frac{a_\lambda-\overline{a}}{\lambda}-a'(y_x)y'_x\right)_{xt}\right|^2|y_x|^2dxdt\\
&+\iint_Q \rho_3^2\left|\left(\frac{a_\lambda-\overline{a}}{\lambda}-a'(y_x)y'_x\right)_x\right|^2|y_{xt}|^2dxdt\\
&+\iint_Q \rho_3^2|(a_\lambda-\overline{a})_t|^2|y'_{xx}|^2dxdt+\iint_Q \rho_3^2|(a_\lambda-\overline{a})_x|^2|y'_{xt}|^2dxdt\\
&+\iint_Q \rho_3^2|a_\lambda-\overline{a}|^2|y'_{xxt}|^2dxdt+\iint_Q \rho_3^2|(a_\lambda-\overline{a})_{xt}|^2|y'_x|^2dxdt\\
&+\iint_Q \rho_3^2\left|\left(\frac{a_\lambda-\overline{a}}{\lambda}-a'(y_x)y'_x\right)_t\right|^2|y_{xx}|^2dxdt\\
&+\iint_Q \rho_3^2\left|\left(\frac{a_\lambda-\overline{a}}{\lambda}-a'(y_x)y'_x\right)_{xt}\right|^2|y_x|^2dxdt\\
&+\iint_Q \rho_3^2\left|\frac{F'_{\lambda,1}-\overline{F}'_1}{\lambda}-D^2_{11}F(y,y_x)y'-D^2_{12}F(y,y_x)y'_x\right|^2|y_t|^2dxdt\\
&+\iint_Q \rho_3^2\left|\frac{F'_{\lambda,2}-\overline{F}'_2}{\lambda}-D^2_{21}F(y,y_x)y'-D^2_{22}F(y,y_x)y'_x\right|^2|y_{xt}|^2dxdt\\
&\left.+\lambda^2\iint_Q \rho_3^2(|y'|^2+|y'_x|^2)(|y'_t|^2+|y'_{xt}|^2)dxdt\right).
\end{aligned}
$$

Using Observation 4.2.8 and Lebesgue's Theorem, we find that

$$
\frac{1}{\lambda}[\mathcal{A}_{1,t}((y,p^1,p^2,f)+\lambda(y',p^{1'},p^{2'},f'))-\mathcal{A}_{1,t}(y,p^1,p^2,f)]\to D\mathcal{A}_{1,t}(y',p^{1'},p^{2'},f').
$$

Similarly

$$
\begin{aligned}
&\|\frac{1}{\lambda}[\mathcal{A}_1((y,p^1,p^2,f)+\lambda(y',p^{1'},p^{2'},f'))(0)-\mathcal{A}_1(y,p^1,p^2,f)(0)]-D\mathcal{A}_1(y',p^{1'},p^{2'},f')(0)\|^2_{H^1_0(I)}\\
&\quad\leqslant C\Bigg(\int_I |[(\frac{1}{\lambda}(a(y_x(0)+\lambda y'_x(0))-a(y_x(0)))-a'(y_x(0))y'_x(0))y_x(0)]_{xx}|^2dx\\
&\quad+\int_I |[(a(y_x(0)+\lambda y'_x(0))-a(y_x(0))-)y_x(0)]_{xx}|^2dx\\
&\quad+\iint_Q \rho^2\left|\left[\frac{F(y+\lambda y',y_x+\lambda y'_x)(0)-F(y,y_x)(0)}{\lambda}-\nabla F(y,y_x)(0)(y',y'_x)(0)\right]_x\right|^2dxdt\Bigg).
\end{aligned}
$$

Using Observation 4.2.8 and Lebesgue's Theorem, we find that

$$
\frac{1}{\lambda}[\mathcal{A}_1((y,p^1,p^2,f)+\lambda(y',p^{1'},p^{2'},f'))(0)-\mathcal{A}_1(y,p^1,p^2,f)(0)]\to D\mathcal{A}_1(y',p^{1'},p^{2'},f')(0).
$$

and finally

$$
\begin{aligned}
&\|\frac{1}{\lambda}[\mathcal{A}_i((y,p^1,p^2,f)+\lambda(y',p^{1'},p^{2'},f'))-\mathcal{A}_2(y,p^1,p^2,f)]-D\mathcal{A}_2(y',p^{1'},p^{2'},f')\|^2_{L^2(\rho^2;Q)}\\
&\quad\leqslant C\Bigg(\iint_Q \rho^2\left|\left(\left(\frac{a'_\lambda-\overline{a}'}{\lambda}-\overline{a}''y'_x\right)y_xp^i_x\right)_x\right|^2dxdt\\
&\quad+\iint_Q \rho^2\left|\left(\left(\frac{a_\lambda-\overline{a}}{\lambda}-\overline{a}'y'_x\right)p^i_x\right)_x\right|^2dxdt\\
&\quad+\iint_Q \rho^2|((a'_\lambda-\overline{a}')y_x{p^{i'}}_x)_x|^2dxdt+\iint_Q \rho^2|((a'_\lambda-\overline{a}')y'_xp^i_x)_x|^2|p^{i'}|^2dxdt\\
&\quad+\iint_Q \rho^2|((a'_\lambda y_x{p^{i'}}_x)_x|^2dxdt\\
&\quad+\iint_Q \rho^2\left|\frac{F'_{\lambda,1}-\overline{F}'_1}{\lambda}-D^2_{11}F(y,y_x)y'p-D^2_{12}F(y,y_x)y'_x\right|^2|p^i|^2dxdt\\
&\quad+\iint_Q \rho^2\left|\frac{(F'_{\lambda,2}-\overline{F}'_2)_x}{\lambda}-(D^2_{21}F(y,y_x)y'p^i)_x-(D^2_{22}F(y,y_x)y'_xp^i)_x\right|^2dxdt\\
&\quad+\iint_Q \rho^2|F'_{\lambda,1}-\overline{F}'_1|^2|p^{i'}|^2dxdt+\iint_Q \rho_3^2|((F'_{\lambda,2}-\overline{F}'_2)p^{i'})_x|^2dxdt\Bigg)
\end{aligned}
$$

Using Observation 4.2.8 and Lebesgue's Theorem, we find that

$$
\frac{1}{\lambda}[\mathcal{A}_i((y,p^1,p^2,f)+\lambda(y',p^{1'},p^{2'},f'))-\mathcal{A}_i(y,p^1,p^2,f)]\to D\mathcal{A}_i(y',p^{1'},p^{2'},f').
$$

Then $\mathcal{A}$ is $G-$differentiable at any $(y,p^1,p^2,f)\in Y$, with a $G-$derivative

$$
\mathcal{A}'(y,p^1,p^2,f)=D\mathcal{A}
$$

Now, we shall prove that the mapping $(y, p^1, p^2, f) \mapsto \mathcal{A}'(y, p^1, p^2, f)$ is continuous from Y into $\mathcal{L}(Y; Z)$. As a consequence, in view of classical results, we will have that $\mathcal{A}$ is not only $G-$differentiable but also $F-$differentiable and C^1.

Thus, let us assume that $(y^n, p^{1,n}, p^{2,n}, f^n) \to (y, p^1, p^2, f)$ in Y and let us check that

$$\|(D\mathcal{A}(y^n, p^{1,n}, p^{2,n}, f^n) - D\mathcal{A}(y, p^1, p^2, f))(y', p^{1'}, p^{2'}, f')\|_Y^2 \leqslant \epsilon_n \|(y', p^{1'}, p^{2'}, f')\|_Y^2 \tag{4.57}$$

for all $(y', p^{1'}, p^{2'}, f') \in Y$, for some $\epsilon_n \to 0$.

The following holds, using the Observation 4.2.8 and Lebesgue's Theorem

$$\begin{aligned}
&\|(D\mathcal{A}_1(y^n, p^{1,n}, p^{2,n}, f^n) - D\mathcal{A}_1(y, p^1, p^2, f))(y', p^{1'}, p^{2'}, f')\|_{L^2(\rho^2;Q)}^2 \\
&\quad \leqslant C\left(\iint_Q \rho^2 |(\overline{a}_n' y_x' y_{n,x})_x - (\overline{a}' y_x' y_x)_x + (\overline{a}_n' y_x')_x - (\overline{a}' y_x')_x|^2 dxdt\right. \\
&\quad + \iint_Q \rho^2 |D_1 F(y^n, y_x^n) - D_1 F(y, y_x)|^2 |y'|^2 dxdt \\
&\quad \left. + \iint_Q \rho^2 |D_2 F(y^n, y_x^n) - D_2 F(y, y_x)|^2 |y_x'|^2 dxdt\right) \\
&\quad \leqslant C\left(\iint_Q \rho^2 |((\overline{a}_n' - \overline{a}') y_x' y_x)_x|^2 dxdt + \iint_Q \rho^2 |(\overline{a}_n' y_x'(y_{n,x} - y_x))_x|^2 dxdt\right. \\
&\quad \left. + \iint_Q \rho^2 |((\overline{a}_n' - \overline{a}') y_x')_x|^2 dxdt + \iint_Q \rho^2 (|y^n - y|^2 + |y_x^n - y_x|^2)(|y'|^2 + |y_x'|^2) dxdt\right) \\
&\quad \leqslant \epsilon_{1,n} \|(y', p^{1'}, p^{2'}, f')\|_Y^2
\end{aligned}$$

where

$$\epsilon_{1,n} := C(1 + \|(y^n, p^{1,n}, p^{2,n}, f^n)\|_Y^2 + \|(y, p^1, p^2, f)\|_Y^2) \|(y^n, p^{1,n}, p^{2,n}, f^n) - (y, p^1, p^2, f)\|_Y^2$$

For the other component, similar arguments lead to the same conclusion.

$$
\begin{aligned}
&\|(D\mathcal{A}_{1,t}(y^n,p^{1,n},p^{2,n},f^n)-D\mathcal{A}_{1,t}(y,p^1,p^2,f))(y',p^{1'},p^{2'},f')\|^2_{L^2(\rho_3^2;Q)}\\
&\leqslant C\Big(\iint_Q \rho_3^2|((\overline{a}'_n-\overline{a}')y'_xy_x)_{xt}|^2dxdt+\iint_Q \rho_3^2|(\overline{a}'_ny'_x(y_{n,x}-y_x))_{xt}|^2dxdt\\
&+\iint_Q \rho_3^2|((\overline{a}'_n-\overline{a}')y'_x)_{xt}|^2dxdt\\
&+\iint_Q \rho_3^2|D^2_{11}F(y^n,y^n_x)y^n_t-D^2_{11}F(y,y_x)y_t|^2|y'|^2dxdt\\
&+\iint_Q \rho_3^2|D^2_{12}F(y^n,y^n_x)y^n_{xt}-D^2_{12}F(y,y_x)y_{xt}|^2|y'|^2dxdt\\
&+\iint_Q \rho_3^2|D^2_{21}F(y^n,y^n_x)y^n_t-D^2_{21}F(y,y_x))y_t|^2|y'_x|^2dxdt\\
&+\iint_Q \rho_3^2|D^2_{22}F(y^n,y^n_x)y^n_{xt}-D^2_{22}F(y,y_x)y_{xt}|^2|y'_x|^2dxdt\\
&+\iint_Q \rho_3^2|F'_{n,1}-\overline{F}'_1|^2|y'_t|^2dxdt+\iint_Q \rho_3^2|F'_{n,2}-\overline{F}'_2|^2|y'_{xt}|^2dxdt\Big)\\
&\leqslant \epsilon_{2,n}\|(y',p^{1'},p^{2'},f')\|^2_Y
\end{aligned}
$$

where $\epsilon_{2,n}\to 0$ as $n\to+\infty$.

Similarly

$$
\begin{aligned}
&\|(D\mathcal{A}_1(y^n,p^{1,n},p^{2,n},f^n)-D\mathcal{A}_1(y,p^1,p^2,f))(y',p^{1'},p^{2'},f')(0)\|^2_{H^1_0(I)}\\
&\leqslant C\int_I|[(a'(y^n_x(0)y^n_x(0)-a'(y_x(0))y_x(0))y'_x(0)]_{xx}|^2dx+\int_I|[(a(y^n_x(0)-a(y_x(0)))y'_x(0)]_{xx}|^2dx\\
&+\int_I|[F'_{n,1}(0)y^{n\prime}(0)-\overline{F}'_1(0)y'(0)]_x|^2dx+\int_I|[F'_{n,2}(0)y^{n\prime}{}_x(0)-\overline{F}'_2(0)y'_x(0)]_x|^2dx\\
&\leqslant \epsilon_{3,n}\|(y',p^{1'},p^{2'},f')\|^2_Y
\end{aligned}
$$

where $\epsilon_{3,n}\to 0$ as $n\to+\infty$.

And

$$
\begin{aligned}
\|(D\mathcal{A}_i(y^n,p^{1,n},p^{2,n},f^n)-D\mathcal{A}_i(y,p^1,p^2,f))(y',p^{1'},p^{2'},f')\|^2_{L^2(\rho^2;Q)}&\\
\leqslant C\epsilon_{i+2,n}\|(y',p^{1'},p^{2'},f')\|^2_Y&
\end{aligned}
$$

where $\epsilon_{i+2,n}\to 0$ as $n\to+\infty$.

This show that (4.57) is satisfied. □

Lemma 4.2.11. *Let $\mathcal{A}$ be the mapping defined by* (4.56). *Then $\mathcal{A}'(0,0,0,0)$ is onto.*

Proof. Let us fix $(G, G_1, G_2, y_0) \in Z$. From Proposition 4.2.5 we know that there exists (y, p^1, p^2, f) satisfying (4.32), (4.33) and (4.5). Consequently, $(y, p^1, p^2, f) \in Y$ and

$$\begin{aligned}\mathcal{A}'(0,0,0,0)(y, p^1, p^2, f) = (&y_t - a(0)y_{xx} + D_1F(0,0)y + D_1F(0,0)y_x - f1_{\mathcal{O}}\\ &+ \frac{1}{\mu_1}p^1 1_{\mathcal{O}_1} + \frac{1}{\mu_2}p^2 1_{\mathcal{O}_2},\\ &- p^1_t - a(0)p^1_{xx} + D_1F(0,0)p^1 - D_2F(0,0)p^1_x - \alpha_1 y 1_{\mathcal{O}_{1,d}},\\ &- p^2_t - a(0)p^2_{xx} + D_1F(0,0)p^2 - D_2F(0,0)p^2_x - \alpha_2 y 1_{\mathcal{O}_{2,d}},\\ &y(0)) = (G, G_1, G_2, y_0)\end{aligned}$$

This end the proof. □

In accordance with Lemmas 4.2.9, 4.2.10 and 4.2.11, we can apply Theorem 1.1.2 and deduce that, there exists $\epsilon > 0$, a mapping $W : B_\epsilon(0) \subset Z \to Y$ such that

$$W(w) \in B_r(0) \ \text{ and } \ \mathcal{A}(W(w)) = w \ , \quad \forall w \in B_\epsilon(0)$$

Taking $(0, -\alpha_1 y_{1,d} 1_{\mathcal{O}_{1,d}}, -\alpha_2 y_{2,d} 1_{\mathcal{O}_{2,d}}, y_0) \in B_\epsilon(0)$ and

$$(y, p^1, p^2, f) = W(0, -\alpha_1 y_{1,d} 1_{\mathcal{O}_{1,d}}, -\alpha_2 y_{2,d} 1_{\mathcal{O}_{2,d}}, y_0) \in Y,$$

we have

$$\mathcal{A}((y, p^1, p^2, f)) = (0, -\alpha_1 y_{1,d} 1_{\mathcal{O}_{1,d}}, -\alpha_2 y_{2,d} 1_{\mathcal{O}_{2,d}}, y_0)$$

thus, we prove that (4.15) is null locally controllable at time $T > 0$.

4.3 Nash equilibrium for (4.1)

Let $f \in L^2(\mathcal{O} \times (0,T))$ be given and let (v^1, v^2) be the associated Nash quasi-equilibrium. For any $s \in \mathbb{R}$ and $w^1, w^2 \in L^2(\mathcal{O}_1 \times (0,T))$, we have

$$\begin{aligned}\langle D_1J_1(f; v^1 + sw^1, v^2), w^2\rangle = \alpha_1 &\iint_{\mathcal{O}_{1,d}\times(0,T)} (y^s - y_{1,d})z^s\,dxdt\\ &+ \mu_1 \iint_{\mathcal{O}_1\times(0,T)} (v^1 + sw^1)w^2\,dxdt \qquad (4.58)\end{aligned}$$

where

$$\begin{cases} y^s_t - (a(y^s_x)y^s_x)_x + F(y^s, y^s_x) = f1_{\mathcal{O}} + (v^1 + sw^1)1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_1} \text{ in } Q,\\ y^s(0,t) = y^s(L,t) = 0 \quad \text{in} \quad (0,T),\\ y^s(0) = y^0 \quad \text{in} \quad I, \end{cases} \qquad (4.59)$$

z^s the derivative of the state y^s with respect to v^1 in the direction w^2, i. e. the solution to

$$\begin{cases} z^s_t - ((a'(y^s_x)y^s_x + a(y^s_x))z^s_x)_x + D_1F(y^s, y^s_x)z^s + D_2F(y^s, y^s_x)z^s_x = w^2 1_{\mathcal{O}_1} \text{ in } Q, \\ z^s(0,t) = z^s(L,t) = 0 \quad \text{in} \quad (0,T), \\ z^s(0) = 0 \quad \text{in} \quad I. \end{cases} \tag{4.60}$$

with $y = y^s|_{s=0}$ and $z = z^s|_{s=0}$, then

$$\langle DJ_1(f; v^1, v^2), w^2\rangle = \alpha_1 \iint_{\mathcal{O}_{1,d}\times(0,T)} (y - y_{1,d})z \; dxdt + \mu_1 \iint_{\mathcal{O}_1\times(0,T)} v^1 w^2 dxdt \tag{4.61}$$

From (4.58) and (4.61) we have

$$\begin{aligned} \langle D_1J_1(f; v^1 + sw^1, v^2) - D_1J_1(f; v^1, v^2), w^2\rangle = \alpha_1 &\iint_{\mathcal{O}_{1,d}\times(0,T)} (y^s - y_{1,d})z^s dxdt \\ - \alpha_1 &\iint_{\mathcal{O}_{1,d}\times(0,T)} (y - y_{1,d})z \; dxdt \\ + s\mu_1 &\iint_{\mathcal{O}_1\times(0,T)} w^1 w^2 dxdt. \end{aligned} \tag{4.62}$$

Let us introduce the adjoint of (4.60)

$$\begin{cases} -\phi^s_t - ((a'(y^s_x)y^s_x + a(y^s_x))\phi^s_x)_x + D_1F(y^s, y^s_x)\phi^s - (D_2F(y^s, y^s_x)\phi^2)_x \\ = \alpha_1(y^s - y_{1,d})1_{\mathcal{O}_{1,d}} \text{ in } Q, \\ \phi^s(0,t) = \phi^s(L,t) = 0 \quad \text{in} \quad (0,T), \\ \phi^s(T) = 0 \quad \text{in} \quad I. \end{cases} \tag{4.63}$$

Multiplying $(4.60)_1$ by ϕ^s in Q, integrating by parts and replacing (4.63), we obtain

$$\begin{aligned} &\iint_Q (z^s_t - ((a'(y^s_x)y^s_x + a(y^s_x))z^s_x)_x + D_1F(y^s, y^s_x)z^s + D_2F(y^s, y^s_x)z^s_x)\phi^s dxdt \\ &\qquad = \iint_Q w^2 1_{\mathcal{O}_1}\phi^s dxdt \\ &\iint_Q (-\phi^s_t - ((a'(y^s_x)y^s_x + a(y^s_x))\phi^s_x)_x + D_1F(y^s, y^s_x)\phi^s - (D_2F(y^s, y^s_x)\phi^2)_x)z^s dxdt \\ &\qquad = \iint_Q w^2\phi^s 1_{\mathcal{O}_1} dxdt \\ &\iint_Q \alpha_1(y^s - y_{1,d})z^s 1_{\mathcal{O}_{1,d}} dxdt = \iint_Q w^2\phi^s 1_{\mathcal{O}_1} dxdt \end{aligned} \tag{4.64}$$

From (4.62) and (4.64) , we have

$$\langle D_1 J_1(f; v^1 + sw^1, v^2) - D_1 J_1(f; v^1, v^2), w^2 \rangle = \iint_{\mathcal{O}_1 \times (0,T)} (\phi^s - \phi) w^2 dxdt$$
$$+ s\mu_1 \iint_{\mathcal{O}_1 \times (0,T)} w^1 w^2 dxdt. \quad (4.65)$$

Notice that

$$\begin{aligned}
-(\phi^s - \phi)_t &- [(a'(y_x^s) y_x^s + a(y_x^s))(\phi_x^s - \phi_x)]_x \\
&- [((a'(y_x^s) - a'(y_x)) y_x^s + a'(y_x)(y^s - y)_x + a(y_x^s) - a(y_x))\phi_x]_x \\
&+ [D_1 F(y^s, y_x^s) - D_1 F(y, y_x^s)]\phi^s + [D_1 F(y, y_x^s) - D_1 F(y, y_x)]\phi^s \\
&+ D_1 F(y, y_x)(\phi^s - \phi) - ([D_2 F(y^s, y_x^s) - D_2 F(y, y_x^s)]\phi^s)_x \\
&- ([D_2 F(y, y_x^s) - D_2 F(y, y_x)]\phi^s)_x - (D_2 F(y, y_x)[\phi^s - \phi])_x \\
&= \alpha_1 (y^s - y) 1_{\mathcal{O}_{1,d}},
\end{aligned}$$

and

$$\begin{aligned}
(y^s - y)_t &- [(a(y_x^s) - a(y_x)) y_x^s + a(y_x)(y_x^s - y_x)]_x + [F(y^s, y_x^s) - F(y, y_x^s)] \\
&+ [F(y, y_x^s) - F(y, y_x)] = sw^1 1_{\mathcal{O}_1}.
\end{aligned}$$

Consequently, the limits

$$\eta = \lim_{s \to 0} \frac{1}{s}(\phi^s - \phi) \quad \text{and} \quad h = \lim_{s \to 0} \frac{1}{s}(y^s - y)$$

exist and satisfy

$$\begin{cases}
-\eta_t - [(a'(y_x) y_x + a(y_x))\eta_x]_x - [(a''(y_x) y_x h_x + 2a'(y_x) h_x)\phi_x]_x \\
+ D_{11}^2 F(y, y_x)\phi h + D_{12}^2 F(y, y_x)\phi h_x + D_1 F(y, y_x)\eta \\
-(D_{21}^2 F(y, y_x)\phi h)_x - (D_{22}^2 F(y, y_x)\phi h_x)_x - (D_2 F(y, y_x)\eta)_x \\
= \alpha h 1_{\mathcal{O}_{1,d}} \text{ in } Q, \\
h_t - [(a'(y_x) y_x + a(y_x)) h_x]_x + D_1 F(y, y_x) h + D_2 F(y, y_x) h_x = w^1 1_{\mathcal{O}_1} \text{ in } Q, \\
\eta(0,t) = \eta(L,t) = 0, \quad h(0,t) = h(L,t) = 0 \quad \text{in } (0,T), \\
\eta(T) = 0, \quad h(0) = 0 \quad \text{in } I.
\end{cases} \quad (4.66)$$

Thus, from (4.65) and (4.66), we deduce that

$$\langle D_1^2 J_1(f; v^1, v^2), (w^1, w^2) \rangle = \iint_{\mathcal{O}_1 \times (0,T)} \eta w^2 dxdt + \mu_1 \iint_{\mathcal{O}_1 \times (0,T)} w^1 w^2 dxdt.$$

In particular, for $w^2 = w^1$, one has

$$\langle D_1^2 J_1(f; v^1, v^2), (w^1, w^1)\rangle = \iint_{\mathcal{O}_1\times(0,T)} \eta w^1 dxdt + \mu_1 \iint_{\mathcal{O}_1\times(0,T)} |w^1|^2 dxdt. \quad (4.67)$$

Let us show that, for some C only depending on I, $\mathcal{O}$, $\mathcal{O}_i$, T, $\mathcal{O}_{i,d}$, α_1, we have

$$\left|\iint_{\mathcal{O}_1\times(0,T)} \eta w^1 dxdt\right| \leqslant C(1 + \|y_0\| + \|f\|_{L^2(\mathcal{O}\times(0,T))})\|w^1\|^2_{L^2(\mathcal{O}_1\times(0,T))} \quad (4.68)$$

We also get the following

$$\begin{aligned}
&\iint_{\mathcal{O}_1\times(0,T)} \eta w^1 dxdt \\
&\quad = \iint_Q (h_t - [(a'(y_x)y_x + a(y_x))h_x]_x + D_1F(y, y_x)h + D_2F(y, y_x)h_x)\eta dxdt \\
&\quad = \iint_Q h(-\eta_t - [(a'(y_x)y_x + a(y_x))\eta_x]_x + D_1F(y, y_x)\eta + (D_2F(y, y_x)\eta)_x dxdt \\
&\quad = \iint_Q h([(a''(y_x)y_xh_x + 2a'(y_x)h_x)\phi_x]_x - D^2_{11}F(y, y_x)\phi h - D^2_{12}F(y, y_x)\phi h_x \\
&\qquad\qquad + (D^2_{21}F(y, y_x)\phi h)_x + (D^2_{22}F(y, y_x)\phi h_x)_x + \alpha h 1_{\mathcal{O}_{1,d}})dxdt \\
&\quad = \iint_Q (a''(y_x)|h_x|^2 y_x\phi_x + 2a'(y_x)|h_x|^2\phi_x - D^2_{11}F(y, y_x)\phi|h|^2 - D^2_{12}F(y, y_x)\phi h_x h \\
&\qquad\qquad - D^2_{21}F(y, y_x)\phi h h_x + D^2_{22}F(y, y_x)\phi|h_x|^2 + \alpha|h|^2 1_{\mathcal{O}_{1,d}})dxdt \\
&\quad \leqslant C\ \left(\int_0^T \|h_{xx}(t)\|^2\|y_x(t)\|\|\phi_x(t)\|dt + \int_0^T \|h_{xx}(t)\|\|h_x(t)\|\|\phi_x(t)\|dt\right. \\
&\qquad\qquad \left. + \int_0^T \|h_x(t)\|\|h(t)\|\|\phi_x(t)\| + \iint_Q |h|^2 dxdt\right) \quad (4.69)
\end{aligned}$$

From (4.63) with $s = 0$, using energy estimates, we have

$$\|\phi_{xx}\|^2_{L^2(Q)} + \|\phi_x\|^2_{L^\infty(0,T;L^2(I))} \leqslant C(\|y\|^2_{L^2(Q)} + \|y_{1,d}\|^2_{L^2(\mathcal{O}_{1,d}\times(0,T))}) \quad (4.70)$$

$$\|h_{xx}\|^2_{L^2(Q)} + \|h_x\|^2_{L^\infty(0,T;L^2(I))} \leqslant C\|w^1\|^2_{L^2(\mathcal{O}_1\times(0,T))} \quad (4.71)$$

as (v^1, v^2) is the Nash quasi-equilibrium, then y have the following regularity

$$\|y\|^2_{L^2(Q)} \leqslant C(\sup_{[0,T]}\|y_x(t)\|^2 + \|y_{xx}\|^2_{L^2(Q)}) \leqslant C(\|f\|^2_{L^2(\mathcal{O}\times(0,T))} + \|y_0\|^2 + \sum_{i=1}^2 \frac{1}{\mu}\|\phi^i\|^2_{L^2(\mathcal{O}_i\times(0,T))}) \quad (4.72)$$

Using (4.69) - (4.72), we have

$$\begin{aligned}
\left|\iint_{\mathcal{O}_1\times(0,T)} \eta w^1 dxdt\right| &\leqslant C\ (\|\phi_x\|_{L^\infty(0,T;L^2)}\|y_x\|_{L^\infty(0,T;L^2)} + \|\phi_x\|_{L^\infty(0,T;L^2)} + 1) \\
&\qquad \cdot (\|h_{xx}\|_{L^2} + \|h_{xx}\|_{L^2}\|h_x\|_{L^\infty(0,T;L^2)}) \\
&\leqslant C\ (\|f\|^2_{L^2(\mathcal{O}\times(0,T))} + \|y_0\|^2 + \|f\|_{L^2(\mathcal{O}\times(0,T))} + \|y_0\| + 1)\|w^1\|^2_{L^2(\mathcal{O}_1\times(0,T))}
\end{aligned}$$

This prove (4.68) in this case.

Taking into account (4.67) and (4.68), we see that

$$\langle D_1^2 J_1(f; v^1, v^2), (w^1, w^1)\rangle \geqslant [\mu_1 - C(\|f\|_{L^2(\mathcal{O}\times(0,T))}, \|y_0\|)] \iint_{\mathcal{O}_1\times(0,T)} |w^1|^2 dxdt.$$

Note that the previous constant C can be chosen independent of μ_1 and μ_2. It is clear that, for sufficiently large μ_1 and μ_2, the couple (v^1, v^2) is a Nash equilibrium in the sense of Definition 4.1.1.

4.4 Additional Commentary

If let us fix an uncontrolled trajectory of (4.1), that is, a sufficiently regular solution to the system

$$\begin{cases} \overline{y}_t - (a(\overline{y}_x)\overline{y}_x)_x + F(\overline{y}, \overline{y}_x) = 0 \text{ in } Q, \\ \overline{y}(0,t) = \overline{y}(L,t) = 0 \quad \text{in} \quad (0,T), \\ \overline{y}(0) = \overline{y}_0 \quad \text{in} \quad I. \end{cases} \tag{4.73}$$

we can prove the same results of the Theorem 4.1.3 and Theorem 4.1.4 with $y(T) = \overline{y}(T)$, following the same steps than the previously result. But, necessarily, the uncontrolled trajectory will be near to the zero, because we will need initials data very smaller.

Chapter 5

Exact Controllability for a Hyperbolic Equation with Non-Local Terms

5.1 Introduction

Let us consider the system

$$\begin{cases} y'' - a(\int_I y dx') y_{xx} = v 1_\omega & \text{in } \ Q, \\ y(0,t) = y(L,t) = 0 & \text{in } \ (0,T), \\ y(x,0) = y_0(x), \quad y'(x,0) = y_1(x) & \text{in } \ I, \end{cases} \tag{5.1}$$

where v is the control and y is the associated state. Let us assume that the real function $a = a(r)$ is of class C^1, possesses bounded derivative and satisfies

$$0 < a_0 \leqslant a(r) \leqslant a_1, \quad \forall r \in \mathbb{R}.$$

Definition 5.1.1. *We say that* (5.1) *is locally exact-controllable at time T if there exists $\epsilon > 0$ such that, for any $(y_0, y_1), (z_0, z_1) \in (H_0^1(I) \cap H^2(I)) \times H_0^1(I)$ with*

$$\|(y_0, y_1)\|_{(H_0^1(I)\cap H^2(I))\times H_0^1(I)} + \|(z_0, z_1)\|_{(H_0^1(I)\cap H^2(I))\times H_0^1(I)} \leqslant \epsilon,$$

there exist controls $v \in L^2(0,T; H^1(\omega))$ such that the associated states y satisfy

$$y(x,T) = z_0(x), \quad y'(x,T) = z_1(x) \ \ \textit{in} \ \ I. \tag{5.2}$$

Our main result in this chapter is the following:

Theorem 5.1.2. *Given $T > 2\sqrt{a_1}\ max\{l_1, L - l_2\}$, under the previous assumptions on $a(\cdot)$, the non-linear system* (5.1) *is locally exact-controllable at time T.*

We can also obtain results for the following system with boundary control

$$\begin{cases} y'' - a(\int_I y dx') y_{xx} = 0 & \text{in } \ Q, \\ y(0,t) = v(t), y(L,t) = 0 & \text{in } \ (0,T), \\ y(x,0) = y_0(x), \quad y'(x,0) = y_1(x) & \text{in } \ I, \end{cases} \tag{5.3}$$

we have the following result

Theorem 5.1.3. *Given $T > 2\sqrt{a_1}L$, under the assumptions on $a(\cdot)$, then, for every $(y_0, y_1), (z_0, z_1) \in H_0^1(I) \times L^2(I)$ small enough, there exist a control $v \in C^1([0,T])$ such that the solution y of (5.3) satisfies (5.2)*

This theorem is a consequence of Theorem 5.1.2.

The proof of Theorem 5.1.2 relies on an application of Schauder's Fixed Point Theorem. We will follow the ideas of [33].

This chapter is organized as follows. In Section 5.2 we will show the development of the fixed point method, this is, look for the suitable spaces for apply this technique and finally prove Theorem 5.1.2 using suitable observability estimates for the hyperbolic equation. In Section 5.3 we will prove observability estimates mentioned in the previous section. In Section 5.4 we will deal the boundary control problem, this is, Theorem 5.1.3 is proved as a consequence of Theorem 5.1.2.

5.2 Description of the Fixed-Point Method

In this section we will describe the Fixed-Point technique that we will use in the proof of Theorem 5.1.2. We proceed in three steps.

Step 1.

Let us fix the initial and final data $\{y_0, y_1\}, \{z_0, z_1\} \in (H_0^1(I) \cap H^2(I)) \times H_0^1(I)$. Given any $\xi \in Z := \{w \in L^\infty(0,T; L^2(I)), w' \in L^\infty(0,T; L^2(I))\}$ we look for a control $v = v(x,t;\xi) \in L^2(0,T; H^1(\omega))$ such that the solution $y = y(x,t,\xi)$ of

$$\begin{cases} y'' - \alpha(t;\xi) y_{xx} = v 1_\omega & \text{in } \ Q, \\ y(0,t) = y(L,t) = 0 & \text{in } \ (0,T), \\ y(x,0) = y_0(x), \quad y'(x,0) = y_1(x) & \text{in } \ I, \end{cases} \tag{5.4}$$

satisfies (5.2), where

$$\alpha(t;\xi) := a(\int_I \xi(x',t)dx').$$

We will use HUM (see [26]).

First, we solve the problem

$$\begin{cases} z'' - \alpha(t;\xi)z_{xx} = 0 & \text{in} \quad Q, \\ z(0,t) = z(L,t) = 0 & \text{in} \quad (0,T), \\ z(x,T) = z_0(x), \quad z'(x,T) = z_1(x) & \text{in} \quad I, \end{cases} \tag{5.5}$$

this system has a unique solution $z = z(x,t;\xi)$ such that

$$z \in L^\infty(0,T;H^2(I)\cap H_0^1(I)),\ z' \in L^\infty(0,T;H_0^1(I)),\ z'' \in L^1(0,T;L^2(I)).$$

Thus

$$z \in C([0,T];H_0^1(I)) \cap C^1([0,T];L^2(I))$$

and therefore

$$z(x,0;\xi) := z_0^\xi \in H_0^1(I), \quad z'(x,0;\xi) := z_1^\xi \in L^2(I). \tag{5.6}$$

Then, for any $\{\phi_0,\phi_1\} \in H_0^1(I)\times L^2(I)$, we solve

$$\begin{cases} \phi'' - \alpha(t;\xi)\phi_{xx} = 0 & \text{in} \quad Q, \\ \phi(0,t) = \phi(L,t) = 0 & \text{in} \quad (0,T), \\ \phi(x,0) = \phi_0(x), \quad \phi'(x,0) = \phi_1(x) & \text{in} \quad I. \end{cases} \tag{5.7}$$

This system has a unique solution $\phi = \phi(x,t;\xi) \in L^2(0,T;H_0^1(I))$.

Now let us consider

$$\begin{cases} \eta'' - \alpha(t;\xi)\eta_{xx} = \phi 1_\omega & \text{in} \quad Q, \\ \eta(0,t) = \eta(L,t) = 0 & \text{in} \quad (0,T), \\ \eta(x,T) = 0, \quad \eta'(x,T) = 0 & \text{in} \quad I, \end{cases} \tag{5.8}$$

we have that

$$\eta \in L^\infty(0,T;H_0^1(I)),\ \eta' \in L^\infty(0,T;L^2(I)),\ \eta'' \in L^2(0,T;H^{-1}(I)).$$

Thus

$$\eta \in C([0,T];L^2(I)) \cap C^1([0,T];H^{-1}(I))$$

and therefore

$$\eta(x,0;\xi) \in L^2(I), \quad \eta'(x,0;\xi) \in H^{-1}(I).$$

Let us define the linear operator $\Lambda_\xi : H_0^1(I) \times L^2(I) \to H^{-1}(I) \times L^2(I)$ by

$$\Lambda_\xi(\phi_0, \phi_1) := (-\eta'(x,0;\xi), \eta(x,0;\xi)) \tag{5.9}$$

we will prove the existence of some $(\phi_0, \phi_1) \in H_0^1(I) \times L^2(I)$ such that

$$\Lambda_\xi(\phi_0, \phi_1) = (-y_1 + z_1^\xi, y_0 - z_0^\xi). \tag{5.10}$$

This conclude the step 1, indeed, if (ϕ_0, ϕ_1) is solution of (5.10), then η, the corresponding solution of (5.8) satisfies

$$\eta'(x,0) = y_1(x) - z_1^\xi(x), \quad \eta(x,0) = y_0(x) - z_0^\xi(x)$$

and if let us define $y := \eta + z$, we get

$$\begin{cases} y'' - \alpha(t;\xi) y_{xx} = \phi 1_\omega & \text{in } \ Q, \\ y(0,t) = y(L,t) = 0 & \text{in } \ (0,T), \\ y(x,0) = y_0(x), \quad y'(x,0) = y_1(x) & \text{in } \ I, \\ y(x,T) = z_0(x), \quad y'(x,T) = z_1(x) & \text{in } \ I. \end{cases} \tag{5.11}$$

In order to solve (5.10) we observe that multiplying the equation (5.8) by ϕ and to integrate by parts, we obtain

$$\int_0^T \int_I (\eta'' - \alpha(t;\xi)\eta_{xx})\phi dx dt = \int_0^T \int_\omega |\phi|^2 dx dt$$
$$\int_I \eta(x,0)\phi_1(x) dx - \int_I \eta'(x,0)\phi_0(x) dx = \int_0^T \int_\omega |\phi|^2 dx dt$$

Then

$$< \Lambda_\xi(\phi_0, \phi_1), (\phi_0, \phi_1) >_{\{H^{-1}(I)\times L^2(I), H_0^1(I)\times L^2(I)\}} = \iint_{\omega\times(0,T)} |\phi|^2 dx dt. \tag{5.12}$$

Let us assume that

$$\|(\phi_0, \phi_1)\|^2_{H_0^1(I)\times L^2(I)} \leqslant C(\|\xi\|_Z) \iint_{\omega\times(0,T)} |\phi|^2 dx dt \tag{5.13}$$

for every $(\phi_0, \phi_1) \in H_0^1(I) \times L^2(I)$.

Let us define the inner product

$$(((\phi_0, \phi_1), (\tilde{\phi}_0, \tilde{\phi_1})))_{H_0^1(I)\times L^2(I)} := \iint_{\omega\times(0,T)} \phi\tilde{\phi} \, dx dt$$

for every $(\phi_0, \phi_1), (\tilde{\phi}_0, \tilde{\phi}_1) \in H_0^1(I) \times L^2(I)$.

Then, using (5.13) we have that

$$|||(\phi_0, \phi_1)|||_{H_0^1(I)\times L^2(I)} := \left(\iint_{\omega\times(0,T)} |\phi|^2 dxdt \right)^{1/2}$$

is a norm and by Cauchy-Schwartz, we get

$$| < \Lambda_\xi(\phi_0, \phi_1), (\tilde{\phi}_0, \tilde{\phi}_1) > | \leqslant |||(\phi_0, \phi_1)||| \, |||(\tilde{\phi}_0, \tilde{\phi}_1)|||.$$

Thus Λ_ξ is linear and continuous.

Now, let $B : (H_0^1(I) \times L^2(I)) \times (H_0^1(I) \times L^2(I)) \to \mathbb{R}$ with

$$B((\phi_0, \phi_1), (\tilde{\phi}_0, \tilde{\phi}_1)) :=< \Lambda_\xi(\phi_0, \phi_1), (\tilde{\phi}_0, \tilde{\phi}_1) > .$$

We have that $B(\cdot, \cdot)$ is a bilinear form, continuous and coercive (using (5.13)). By Lax-Milgram theorem for every $(\gamma_0, \gamma_1) \in H^{-1}(I) \times L^2(I)$, exist a unique $(\hat{\phi}_0, \hat{\phi}_1) \in H_0^1(I) \times L^2(I)$ such that

$$B((\hat{\phi}_0, \hat{\phi}_1), (\phi_0, \phi_1)) :=< (\gamma_0, \gamma_1), (\phi_0, \phi_1) >$$

then

$$\Lambda_\xi(\hat{\phi}_0, \hat{\phi}_1) = (\gamma_0, \gamma_1).$$

This is, the equation (5.10) has a unique solution

$$(\hat{\phi}_0, \hat{\phi}_1) \in H_0^1(I) \times L^2(I)$$

and the function $v(x, t; \xi) := \hat{\phi}(x, t; \xi)$ is the desired control such that the solution y of (5.4) satisfied (5.2).

Step 2.

We have defined a unique control $v(x, t; \xi) \in L^2(0, T; H_0^1(I))$ for system (5.4) and (5.2) for every $\xi \in Z$. The solution y of (5.4) belong to

$$W := \{y \in L^\infty(0, T; H^2(I) \cap H_0^1(I)); y' \in L^\infty(0, T; H_0^1(I)), y'' \in L^1(0, T; L^2(I))\}.$$

Indeed, we have the estimates

$$
\begin{aligned}
\int_I [y'' - \alpha(t;\xi) y_{xx}] y' dx &= \int_I v 1_\omega y' dx \\
\frac{1}{2}\frac{d}{dt}|y'(t)|^2_{L^2(I)} + \frac{1}{2}\frac{d}{dt}\left(\alpha(t;\xi)|y_x(t)|^2_{L^2(I)}\right) &= \int_I v 1_\omega y' dx \\
&\quad + \frac{1}{2} a'\left(\int_I \xi(x',t)dx'\right)\left(\int_I \xi'(x',t)dx'\right)|y_x(t)|^2_{L^2(I)} \\
\frac{1}{2}\frac{d}{dt}|y'(t)|^2_{L^2(I)} + \frac{1}{2}\frac{d}{dt}\left(\alpha(t;\xi)|y_x(t)|^2_{L^2(I)}\right) &\leqslant \int_I |v 1_\omega||y'| dx + \frac{M}{2}\|\xi\|_Z |y_x(t)|^2_{L^2(I)} \\
\frac{d}{dt}\left(|y'(t)|^2_{L^2(I)} + \alpha(t;\xi)|y_x(t)|^2_{L^2(I)}\right) &\leqslant 2\int_I |v 1_\omega||y'| dx + M\|\xi\|_Z |y_x(t)|^2_{L^2(I)} \\
\frac{d}{dt}\left(|y'(t)|^2_{L^2(I)} + \alpha(t;\xi)|y_x(t)|^2_{L^2(I)}\right) &\leqslant |v(t)|^2_{L^2(\omega)} + |y'(t)|^2_{L^2(I)} + M\|\xi\|_Z |y_x(t)|^2_{L^2(I)}
\end{aligned}
$$

then

$$
\frac{d}{dt}\left(|y'(t)|^2_{L^2(I)} + \alpha(t;\xi)|y_x(t)|^2_{L^2(I)}\right) \leqslant C_\xi(|y'(t)|^2_{L^2(I)} + \alpha(t;\xi)|y_x(t)|^2_{L^2(I)}) + |v|^2_{L^2(\omega)}
$$

where $C_\xi := \max\{1, M\|\xi\|_Z / a_0\}$, by Gronwall's inequality

$$
|y'(t)|^2_{L^2(I)} + a_0|y_x(t)|^2_{L^2(I)} \leqslant \left(a_1|y_1|^2_{H^1_0(I)} + |y_1|^2_{L^2(I)} + \int_0^T |v(t)|^2_{L^2(\omega)} dt\right) e^{C_\xi T}.
$$

This is

$$
\|y'\|^2_{L^\infty(0,T;L^2(I))} + \|y\|^2_{L^\infty(0,T;H^1_0(I))} \leqslant \left(a_1|y_0|^2_{H^1_0(I)} + |y_1|^2_{L^2(I)} + \int_0^T |v(t)|^2_{L^2(\omega)} dt\right) e^{C_\xi T} \tag{5.14}
$$

$$
\begin{aligned}
\int_I [y'' - \alpha(t;\xi) y_{xx}](-y'_{xx}) dx &= \int_I v 1_\omega (-y'_{xx}) dx \\
\frac{1}{2}\frac{d}{dt}|y'_x(t)|^2_{L^2(I)} + \frac{1}{2}\frac{d}{dt}\left(\alpha(t;\xi)|y_{xx}(t)|^2_{L^2(I)}\right) &= \int_I v_x 1_\omega y'_x dx \\
&\quad + \frac{1}{2} a'\left(\int_I \xi(x',t)dx'\right)\left(\int_I \xi'(x',t)dx'\right)|y_{xx}(t)|^2_{L^2(I)} \\
\frac{1}{2}\frac{d}{dt}|y'_x(t)|^2_{L^2(I)} + \frac{1}{2}\frac{d}{dt}\left(\alpha(t;\xi)|y_{xx}(t)|^2_{L^2(I)}\right) &\leqslant \int_I |v_x 1_\omega||y'_x| dx + \frac{M}{2}\|\xi\|_Z |y_{xx}(t)|^2_{L^2(I)} \\
\frac{d}{dt}\left(|y'_x(t)|^2_{L^2(I)} + \alpha(t;\xi)|y_{xx}(t)|^2_{L^2(I)}\right) &\leqslant 2\int_I |v_x 1_\omega||y'_x| dx + M\|\xi\|_Z |y_{xx}(t)|^2_{L^2(I)} \\
\frac{d}{dt}\left(|y'_x(t)|^2_{L^2(I)} + \alpha(t;\xi)|y_{xx}(t)|^2_{L^2(I)}\right) &\leqslant |v_x(t)|^2_{L^2(\omega)} + |y'_x(t)|^2_{L^2(I)} + M\|\xi\|_Z |y_{xx}(t)|^2_{L^2(I)}
\end{aligned}
$$

then

$$
\frac{d}{dt}\left(|y'_x(t)|^2_{L^2(I)} + \alpha(t;\xi)|y_{xx}(t)|^2_{L^2(I)}\right) \leqslant C_\xi(|y'_x(t)|^2_{L^2(I)} + \alpha(t;\xi)|y_{xx}(t)|^2_{L^2(I)}) + |v_x|^2_{L^2(\omega)}
$$

by Gronwall's inequality

$$|y'_x(t)|^2_{L^2(I)} + a_0|y_{xx}(t)|^2_{L^2(I)} \leqslant \left(a_1|y_0|^2_{H^2(I)} + |y_1|^2_{H^1_0(I)} + \int_0^T |v_x(t)|^2_{L^2(\omega)}dt \right) \; e^{C_\xi T}.$$

This is

$$\|y'\|^2_{L^\infty(0,T;H^1_0(I))}+\|y\|^2_{L^\infty(0,T;H^2(I))} \leqslant \left(a_1|y_0|^2_{H^2(I)} + |y_1|^2_{H^1_0(I)} + \int_0^T |v_x(t)|^2_{L^2(\omega)}dt \right) \; e^{C_\xi T}. \tag{5.15}$$

Thus

$$\begin{aligned}\|y''\|^2_{L^1(0,T;L^2(I))} &\leqslant 2a_1^2\|y_{xx}\|^2_{L^1(0,T;L^2(I))} + 2\|v1_\omega\|^2_{L^1(0,T;L^2(I))} \\ &\leqslant 2a_1^2 \left(a_1|y_0|^2_{H^2(I)} + |y_1|^2_{H^1_0(I)} + \int_0^T |v_x(t)|^2_{L^2(\omega)}dt \right) \; e^{C_\xi T} + 2\int_0^T |v(t)|^2_{L^2(\omega)}dt.\end{aligned}$$

Then, from (5.14) and (5.15) we have

$$\begin{aligned}&\|y\|^2_{L^\infty(0,T;H^2(I)\cap H^1_0(I))} + \|y'\|^2_{L^\infty(0,T;H^1_0(I))} + \|y''\|^2_{L^1(0,T;L^2(I))} \\ &\leqslant C(a_0,a_1) \left(|y_0|^2_{H^2(I)} + |y_1|^2_{H^1_0(I)} + \int_0^T |v(t)|^2_{L^2(\omega)}dt + \int_0^T |v_x(t)|^2_{L^2(\omega)}dt \right) \; e^{C_\xi T} \\ &+ 2\int_0^T |v(t)|^2_{L^2(\omega)}dt.\end{aligned} \tag{5.16}$$

Therefore, we have constructed a nonlinear operator $K : Z \to Z$ such that $K(\xi) := y(x,t;\xi)$, where y is the solution of (5.4) and satisfies (5.2) with the control function $v(x,t;\xi) \in L^2(0,T;H^1_0(I))$ defined above.

Step 3.

We will look for estimates for the control. We know that $v = \phi$ is solution of (5.7), then

$$\begin{aligned}\int_I \phi''\phi' dx - \int_I a(\int_I \xi dx')\phi_{xx}\phi' dx = 0 \\ \frac{d}{dt}|\phi'(t)|^2_{L^2(I)} + \frac{d}{dt}\left(a(\int_I \xi dx')|\phi_x(t)|^2_{L^2(I)} \right) = a'\left(\int_I \xi dx'\right)\left(\int_I \xi' dx'\right)|\phi_x(t)|^2_{L^2(I)} \\ \frac{d}{dt}\left(|\phi'(t)|^2_{L^2(I)} + a(\int_I \xi dx')|\phi_x(t)|^2_{L^2(I)} \right) \leqslant (M/a_0)\|\xi'\|_Z a(\int_I \xi dx')|\phi_x(t)|^2_{L^2(I)}\end{aligned}$$

using Gronwall's inequality

$$|\phi'(t)|^2_{L^2(I)} + a_0|\phi_x(t)|^2_{L^2(I)} \leqslant (a_1|\phi_0|^2_{H^1_0(I)} + |\phi_1|^2_{L^2(I)}) \; e^{(2M/a_0)\|\xi\|_Z}$$

then

$$|\phi_x(t)|^2_{L^2(\omega)} \leqslant C(a_0,a_1)(|\phi_0|^2_{H^1_0(I)} + |\phi_1|^2_{L^2(I)}) \; e^{(2M/a_0)\|\xi\|_Z}. \tag{5.17}$$

Also, using (5.12), we get

$$\int_0^T |\phi(t)|^2_{L^2(\omega)}dt = \int_I y_0(x)\phi_1(x)dx - \int_I y_1(x)\phi_0(x)dx$$
$$\leqslant C_0(y_0,y_1,z_0,z_1)(\|(\phi_0,\phi_1)\| + \|(\phi(T),\phi'(T))\|) \qquad (5.18)$$

where $C_0(y_0,y_1,z_0,z_1) := max\{\|(y_0,y_1)\|_{L^2(I)\times H^{-1}(I)}, \|(z_0,z_1)\|_{L^2(I)\times H^{-1}(I)}\}$.
Using the time-reversibility of the equation satisfied by ϕ and the observability inequality, we deduce

$$(\|(\phi_0,\phi_1)\| + \|(\phi(T),\phi'(T))\|)^2 \leqslant 4C(\|\xi\|_Z)\int_0^T |\phi(t)|^2_{L^2(\omega)}dt. \qquad (5.19)$$

Combining (5.18) and (5.19) we get

$$\|(\phi_0,\phi_1)\| + \|(\phi(T),\phi'(T))\| \leqslant 4C_0(y_0,y_1,z_0,z_1)C(\|\xi\|_Z). \qquad (5.20)$$

Again, combining (5.18) and (5.20) we obtain

$$\int_0^T |\phi(t)|^2_{L^2(\omega)}dt \leqslant C(y_0,y_1,z_0,z_1)C(\|\xi\|_Z). \qquad (5.21)$$

Finally from (5.17), (5.21) in (5.16) we conclude that

$$\|y\|^2_{L^\infty(0,T;H^2(I)\cap H_0^1(I))} + \|y'\|^2_{L^\infty(0,T;H_0^1(I))} + \|y''\|^2_{L^1(0,T;L^2(I))} \leqslant C(a_0,a_1,y_0,y_1,z_0,z_1)C(\|\xi\|_Z) \qquad (5.22)$$

from the above, we see that the operator K sends bounded sets of Z into bounded sets of W.
This fact, combined with the compactness of embedding (in view of the Simon's Compactness Theorem, see [30])

$$W \hookrightarrow Z$$

allows us to prove both the continuity of K from Z to Z and the fact that K maps bounded sets of Z into relatively compact sets of itself, for initial and final data sufficiently small. Therefore the operator K is compact.
We can use Theorem 1.1.3 to complete the proof.

5.3 Observability Estimates for (5.7)

The aim of this section is to prove the following observability result for system (5.7).

Theorem 5.3.1. *If $T > 2\sqrt{a_1}\ max\{l_1, L - l_2\}$, there exist two positive constants $A, B > 0$ such that*

$$\|\phi_0\|^2_{H^1_0(I)} + \|\phi_1\|^2_{L^2(I)} \leqslant A\ e^{B\|\xi\|_Z} \iint_{\omega\times(0,T)} |\phi|^2 dxdt \tag{5.23}$$

for every ϕ solution of (5.7) with initial data $\{\phi_0, \phi_1\} \in H^1_0(I) \times L^2(I)$.

Proof. We define $\gamma(t) := \displaystyle\int_0^t \frac{dr}{\sqrt{\alpha(r;\xi)}}$, then $\gamma|_{[0,T]} : [0,T] \to [0, \gamma(T)]$ is bijective, whence

$$\gamma^{-1}(t) := (\gamma|_{[0,T]})^{-1}(t) = \int_0^t \sqrt{\alpha((\gamma_{[0,T]})^{-1}(r);\xi)}dr.$$

For any $x_0 \in I = (0, L)$, we denote

$$\tau_1(x_0) := \{(x,t) \in (0,x_0)\times(0,T); t \in (\gamma^{-1}(x_0 - x), \gamma^{-1}(\gamma(T) - (x_0 - x)))\},$$

$$\tau_2(x_0) := \{(x,t) \in (x_0,L)\times(0,T); t \in (\gamma^{-1}(x - x_0), \gamma^{-1}(\gamma(T) - (x - x_0)))\}$$

and

$$\tau(x_0) := \tau_1(x_0) \cup \tau_2(x_0).$$

We observe that due to finite speed of propagation in system (5.7) and the characteristic curves: $\gamma(t) + x = cte$ and $\gamma(t) - x = cte$ for ϕ, we have

$$\phi = \psi \ \text{ in } \ \tau(x_0), \tag{5.24}$$

where $\psi = \psi(x,t;\xi)$ is the solution of

$$\begin{cases} \psi_{xx} - \dfrac{1}{\alpha(t;\xi)}\psi'' = 0 & \text{in} \quad (0,T)\times I, \\ \psi(x,0) = \psi(x,T) = 0 & \text{in} \quad I, \\ \psi(x_0,t) = \phi(x_0,t), \quad \psi_x(x_0,t) = \phi_x(x_0,t) & \text{in} \quad (0,T). \end{cases} \tag{5.25}$$

System (5.25) is hyperbolic equation where the roles of the time and space variables has been interchanged. It is an evolution equation with respect to x.
We can apply estimates of the energy for the system (5.25) and we get

$$\|\psi_x(x)\|^2_{L^2(0,T)} + \frac{1}{a_1}\|\psi'(x)\|^2_{L^2(0,T)} \leqslant \left(\frac{1}{a_0}\|\psi'(x_0)\|^2_{L^2(0,T)} + \|\psi_x(x_0)\|^2_{L^2(0,T)}\right) e^{2M\|\xi\|_Z L/(a_0)^{3/2}}. \tag{5.26}$$

Combining (5.24) and (5.26) we get

$$\iint_{\tau(x_0)} |\phi_x|^2 dxdt \leqslant Ce^{2M\|\xi\|_Z L/(a_0)^{3/2}}(\|\phi'(x_0)\|^2_{L^2(0,T)} + \|\phi_x(x_0)\|^2_{L^2(0,T)}). \tag{5.27}$$

Now, integrating (5.27) with respect to those $x_0 \in \omega$ for which the time T satisfies $\gamma(T) > 2\,\max\{x_0, L - x_0\}$ we get

$$\iint_{I\times(t_1,t_2)} |\phi_x|^2 dxdt \leqslant Ce^{2M\|\xi\|_Z L/(a_0)^{3/2}} \left(\iint_{\omega\times(0,T)} |\phi'|^2 dxdt + \iint_{\omega\times(0,T)} |\phi_x|^2 dxdt \right) \tag{5.28}$$

where $t_1 := \max\{\gamma^{-1}(l_1), \gamma^{-1}(L - l_2)\}$ and $t_2 := T - \max\{\gamma^{-1}(l_1), \gamma^{-1}(L - l_2)\}$. We know that $\phi \in L^2(0,T; H^1(\omega)) = L^2(0,T; H^1(l_1, l_2))$, then we deduce that $\phi'' \in L^2(0,T; H^{-1}(l_1, l_2))$ with

$$\|\phi''\|_{L^2(0,T;H^{-1}(l_1,l_2))} \leqslant C\|\phi\|_{L^2(0,T;H^1(l_1,l_2))}.$$

Then, by interpolation we obtain that $\phi' \in L^2(0,T; L^2(l_1, l_2))$ with

$$\|\phi'\|_{L^2(0,T;L^2(l_1,l_2))} \leqslant C\|\phi\|_{L^2(0,T;H^1(l_1,l_2))}. \tag{5.29}$$

Combining (5.29) and (5.28) we get

$$\iint_{I\times(t_1,t_2)} |\phi_x|^2 dxdt \leqslant Ce^{2M\|\xi\|_Z L/(a_0)^{3/2}} \left(\iint_{\omega\times(0,T)} |\phi|^2 dxdt + \iint_{\omega\times(0,T)} |\phi_x|^2 dxdt \right). \tag{5.30}$$

Now, multiplying by $r(t)\phi$ in the equation (5.7) and integrating by parts in $I\times(t_1, t_2)$ we get

$$\begin{aligned}\int_{t_1}^{t_2} r(t)\|\phi'(t)\|^2_{L^2(I)} = &-\int_{t_1}^{t_2} r'(t)\int_I \phi(t)\phi'(t)dxdt + \left[\int_I r(t)\phi(t)\phi'(t)dx\right]_{t_1}^{t_2} \\ &-\int_{t_1}^{t_2} \alpha(t;\xi)\|\phi_x(t)\|^2_{L^2(I)}dt.\end{aligned}$$

Choosing $r \in C^1([t_1, t_2])$ such that $r(t_1) = r(t_2) = 0$, $r(t) = 1$, $\forall t \in [t_1', t_2']$ with $t_1' = t_1 + \dfrac{t_2 - t_1}{3}$, $t_2' = t_2 - \dfrac{t_2 - t_1}{3}$, $\dfrac{|r'|^2}{r} \in L^\infty(t_1, t_2)$, we get

$$\int_{t_1'}^{t_2'} \|\phi'(t)\|^2_{L^2(I)}dt \leqslant C(\|\xi\|_Z)\int_{t_1}^{t_2} \|\phi_x(t)\|^2_{L^2(I)}dt. \tag{5.31}$$

Combining (5.31) and (5.30) we get

$$\int_{t_1'}^{t_2'} (\|\phi_x(t)\|_{L^2(I)} + \|\phi'(t)\|^2_{L^2(I)})dt \leqslant C(\|\xi\|_Z)\left(\iint_{\omega\times(0,T)} (|\phi|^2 + |\phi_x|^2)dxdt\right). \tag{5.32}$$

By the estimates the energy and the time-reversibility of system (5.7) we have

$$(t_2' - t_1')(\|\phi_0\|^2_{H_0^1(I)} + \|\phi_1\|^2_{L^2(I)}) \leqslant Ce^{M\|\xi\|_Z t_2'/a_0}\int_{t_1'}^{t_2'} (\|\phi_x(t)\|_{L^2(I)} + \|\phi'(t)\|^2_{L^2(I)})dt. \tag{5.33}$$

Combining (5.32) and (5.33) we obtain easily (5.23). □

5.4 Boundary Control

Given any $\epsilon > 0$ let us define the extended domain $\tilde{I} := (-\epsilon, L)$ and $\omega := (-\epsilon, 0)$. Let us extend our initial and final data $(y_0, y_1), (z_0, z_1) \in H_0^1(I) \times L^2(I)$ by zero outside of I to define

$$\tilde{y}_i := \begin{cases} y_i & \text{in } I, \\ 0 & \text{in } (-\epsilon, 0) \end{cases}, \tilde{z}_i := \begin{cases} z_i & \text{in } I, \\ 0 & \text{in } (-\epsilon, 0) \end{cases}, \quad i = 0, 1.$$

Then

$$(\tilde{y}_0, \tilde{y}_1), (\tilde{z}_0, \tilde{z}_1) \in H_0^1(\tilde{I}) \times L^2(\tilde{I}).$$

Since $T > 2\sqrt{a_1}L$, analogously to the proof of Theorem 5.1.2, we deduce that there exists a control $\tilde{v} \in L^2(0, T; H^1(\omega))$ such that the solution $\tilde{y}$ of

$$\begin{cases} \tilde{y}'' - a(\int_I \tilde{y} dx')\tilde{y}_{xx} = \tilde{v} 1_\omega & \text{in } \tilde{I} \times (0, T), \\ \tilde{y}(-\epsilon, t) = \tilde{y}(L, t) = 0 & \text{in } (0, T), \\ \tilde{y}(x, 0) = \tilde{y}_0(x), \quad \tilde{y}'(x, 0) = \tilde{y}_1(x) & \text{in } \tilde{I}, \end{cases}$$

satisfies

$$\tilde{y}(T) = \tilde{z}_0, \quad \tilde{y}'(T) = \tilde{z}_1 \quad \text{in } \tilde{I}.$$

Let be $y = \tilde{y}|_{I \times (0,T)}$ and $v(t) = \tilde{y}(0, t)$. Then y satisfies clearly (5.3) and (5.2). Therefore the control v answers to the question and since $\tilde{y} \in C([0, T]; H^2(\tilde{I}) \cap H_0^1(\tilde{I})) \cap C^1([0, T]; H_0^1(\tilde{I}))$, we deduce that, in particular, $v \in C^1([0, T])$.

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