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GENERALIZED ABEL-PRYM MAPS

Niterói
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Orientadora: Juliana Coelho

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Ata dos trabalhos finais da Comissão Examinadora da Tese de Doutorado em Matemática apresentada por Kelyane Barboza Abreu

Aos cinco dias do mês de abril de dois mil e dezenove, reuniram-se no auditório da Pós-Graduação em Matemática da Universidade Federal Fluminense, os membros da Comissão Examinadora constituída pelos Professores Juliana Coelho Chaves, da Universidade Federal Fluminense; Marco Pacini, da Universidade Federal Fluminense; Giuseppe Borrelli, da Universidade Federal Fluminense; Thiago Fassarella do Amaral, da Universidade Federal Fluminense; Eduardo de Sequeira Esteves, do Instituto Nacional de Matemática Pura e Aplicada e Anita Maria Rojas Rodriguez, da Universidad de Chile, para prova pública de defesa da tese intitulada **"GENERALIZED ABEL-PRYM MAPS"**, apresentada pela Doutoranda Kelyane Barboza de Abreu. A defesa da tese atende às exigências contidas no Regulamento Específico do Curso de Doutorado em Matemática da Universidade Federal Fluminense. A tese foi elaborada sob a orientação da Professora Juliana Coelho Chaves. A Doutoranda Kelyane Barboza Abreu fez a exposição de seu trabalho durante 50 minutos, iniciando às 11h e concluindo às 11h50min. A seguir, respondeu as questões formuladas pelos integrantes da Comissão Examinadora. Terminada a arguição, realizou-se a reunião da Comissão Examinadora, que apresentou parecer no sentido da aprovação da Doutoranda Kelyane Barboza de Abreu, considerando-se o trabalho apresentado e a forma com que se houve na apresentação da defesa do mesmo. Para constar, foi lavrada a presente ata, que vai assinada pela Secretária Administrativa da Coordenação de Pós-Graduação em Matemática, pelos membros da Banca Examinadora e pela Doutoranda.

Niterói, 05 de abril de 2019.

Prof.ª Juliana Coelho Chaves
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(Secretária)

À memória de minha avó, Maria do Carmo, por sempre ter acreditado em mim e me motivado a ser melhor. À memória de Pedro Coelho, que sempre foi reflexo de luz e pureza.

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"A gente nasce para se reinventar"

Resumo

Seja C uma curva suave projetiva não racional sobre o corpo complexo $\mathbb{C}$ e seja $J(C)$ sua Jacobiana principalmente polarizada. Se A é uma subvariedade abeliana de $J(C)$, definimos o mapa de Abel-Prym generalizado $\varphi_A : C \to A$ como a composição do mapa de Abel com o mapa norma de A. O objetivo deste trabalho é entender o grau destes mapas em alguns casos particulares da variedade abeliana A. Primeiro, nós mostramos alguns resultados a respeito do mapa transposto e do mapa de Abel-Prym generalizado. Em seguida, nós discutimos rapidamente sobre a decomposição isotípica de uma variedade abeliana e consideramos esta decomposição no caso particular de $J(C)$. Finalmente, mostramos alguns resultados sobre o grau do mapa φ_A no caso em que A é uma das componentes da decomposição isotípica de $J(C)$ e aplicamos estes resultados em quatro exemplos.

Palavras-chave: Variedades abelianas; Mapa de Abel-Prym generalizado; Variedade Prym-Tyurin.

Abstract

Let C be a smooth non rational projective curve over the complex field $\mathbb{C}$ and let $J(C)$ be its principally polarized Jacobian. If A is an abelian subvariety of $J(C)$, we define the generalized Abel-Prym map $\varphi_A : C \to A$ to be the composition of the Abel map with the norm map of A. The goal of this work is to understand the degree of these maps in some particular cases of the abelian variety A. At first, we show some results about the transpose map and the generalized Abel-Prym map. Then, we quickly discuss the isotypical decomposition of an abelian variety and consider this decomposition on particular case of $J(C)$. Finally, we show some results about the degree of the map φ_A in the case where A is one of the components of the isotypical decomposition of $J(C)$ and apply these results in four examples.

Keywords: Abelian varieties; Generalized Abel-Prym map; Prym-Tyurin variety.

Contents

CHAPTER 1

Introduction

Let C be a smooth projective curve of genus $g(C)$ over the field of complex numbers. Its Jacobian variety $J(C)$ is an abelian variety of dimension $g(C)$, that is, a complex torus along with a polarization Θ, which is a codimension-1 subvariety of $J(C)$ satisfying some properties (cf. Section 2.2 - 2.3). Given a point p in C, we have the morphism $\alpha_p : C \to J(C)$ called Abel-Jacobi map or simply, Abel map. This map is an embedding, and thus, is a canonical way of puting the curve C inside a variety with a natural group structure (cf. Remark 12). The Jacobian varieties are the best-known examples of abelian varieties. A classic result of algebraic geometry on a curve and its Jacobian variety is the Torelli's Theorem. This theorem states that a smooth projective curve C is uniquely determined by its Jacobian variety $J(C)$ (cf. [3, Torelli's Theorem 11.1.7]), giving an injective map between the moduli of smooth projective curves of genus $g(C)$ and the moduli of principally polarized abelian varieties of dimension $g(C)$.

Let A be an abelian subvariety of $J(C)$. We define the map $\varphi_A : C \to A$ to be the composition of the Abel map with the norm map of A (see Section 3.2). The goal of our work is to understand the degree these maps. First, let's consider a classical case of abelian varieties, the Prym varieties. They were introduced by German mathematician Friedrich Emil Fritz Prym. In their original form, they are abelian subvarieties of $J(C)$ associated with a degree-2 cover of smooth curves. The study of these varieties appeared initially in the works of Riemann and they were well studied by Wirtinger in 1895. This theory became dormant for a long time, until they were again studied by David Mumford in 1970. In the case of Prym varieties the degree of map φ_A is known (see Proposition 24).

On any abelian subvariety A of $J(C)$ there exists a natural polarization given by the restriction of the theta divisor Θ to A. In general this restriction is not a multiple of a principal polarization. When it is, we say A is a Prym-Tyurin variety for C or a generalized Prym variety. When A is a Prym-Tyurin variety for C, the map φ_A is called Abel-Prym map.

There are few cases in the literature where the degree of φ_A is calculated. In [4] and [11] we have two examples where this degree is found. In the first work, Brambila-Paz, Gómez-González and Pioli consider a morphism $f : C \to C'$ of degree k, of smooth projective curves. By [4, Theorem 1.1] there exists a Prym-Tyurin variety A for C and, by [4, Theorem 1.2] φ_A is birational onto its image when $g(C') > 2$ and $k \neq 2$. In [11], Lange, Recillas and Rojas constructed a Prym-Tyurin variety A of exponent 3 for a family of curves C and a given correspondence D and showed that the Abel-Prym map φ_A is an embedding. Note that in both cases, the degree of φ_A is one.

The degree of the so called generalized Abel-Prym maps φ_A seems to be rather difficult to understand in full generality. As we will see in Chapter 3 and 4 this problem becomes less difficult in the case where A is a Prym-Tyurin variety for C which is a component of the isotypical decomposition of $J(C)$ (see Chapter 4 for more on this decomposition).

In this sense, we study four examples. In the two initial examples, we consider actions of S_3 and $\mathbb{Z}_2$ on a curve of genus 3. In both cases, the component of the decomposition of dimension 2 is not Prym-Tyurin and in this case we find it difficult to obtain information about the degree of generalized Abel-Prym map. However, in the second case we were able to show that the degree of this map is at least 2. In the third example we consider an action of D_4 on a curve of genus 4. In this example all the components of the decomposition are Prym-Tyurin, and we obtained the degrees of the generalized Abel-Prym maps. Finally, the last example is special because it provides us with a family of generalized Abel-Prym maps of degree 2.

Our work has been divided as follows. Chapter 2 is a summary of known definitions and results concerning abelian varieties, Jacobian varieties and Prym-Tyurin varieties. They were taken from chapters 1-5 and 11-12 of [3]. Let (A, θ) be an abelian variety and let $\varphi : C \to A$ be a morphism. Then, we have the morphism $\tilde{\varphi} : J(C) \to A$ (cf Theorem 13), and we define the transpose map of this morphism, $\tilde{\varphi}^t$ (cf. Definition 25). Chapter 3 is divided in two sections. In the first, we prove some results about the composition $\tilde{\varphi} \circ \tilde{\varphi}^t$ and the degree of the polarization θ. In the second section we consider the generalized Abel-Prym maps and find an upper bound for the degree of this map. When A is a Prym-Tyurin variety this upper bound coincides with its exponent. In Chapter 4 we recall the definition of group actions and we briefly describe the isotypical decomposition of an abelian variety (A, θ) given by these actions. We consider the decompositions for Jacobian variety and prove some interesting results about the components of these decompositions. Finally, in Chapter 5, we apply the results of the previous chapters in some examples to obtain the degree of generalized Abel-Prym maps.

CHAPTER 2

Background

In this chapter we present some definitions and results that we use throughout our work. They were taken from Chapters 1-5 and 11-12 of [3].

2.1 Complex tori

Let V be a complex vector space of dimension g and fix $\alpha_1, ..., \alpha_g, \beta_1, ...\beta_g \in V$ linearly independent over $\mathbb{R}$. Let $\Lambda = \{m_1\alpha_1 + ... + m_g\alpha_g + n_1\beta_1 + ... + n_g\beta_g; \;\; m_i, n_i \in \mathbb{Z}\}$ be a lattice in V, that is, a discrete additive subgroup of V of rank $2g$.

The quotient $T = V/\Lambda$ is called a *complex torus* of dimension g. Note that T is a compact, connected complex manifold, endowed with a group structure. In order to describe T we choose a basis $e_1, ..., e_g$ for V and write the elements $\{\alpha_1, ..., \alpha_g, \beta_1, ...\beta_g\}$ in terms of this basis. Thus, we have

$$\alpha_i = \sum_{j=1}^{g} \lambda_{ji} e_j \;\text{ and }\; \beta_i = \sum_{j=1}^{g} \tilde{\lambda}_{ji} e_j$$

with λ_{ji} and $\tilde{\lambda}_{ji} \in \mathbb{C}$, and the matrix

$$\Pi = ((\lambda_{ji})(\tilde{\lambda}_{ji}))$$

in $M(g \times 2g, \mathbb{C})$ is called a period matrix for T.

There are two distinguished types of holomorphic maps between complex tori, namely homomorphisms and translations. A homomorphism between complex tori is a holomorphic function which preserves the group structure. The translation by an element $x_0 \in T$ is defined to be the holomorphic map $t_{x_0} : x \longmapsto x + x_0$. Under addition the set of homomorphisms of T into T' forms an abelian group denoted by $Hom(T;T')$, where T and T' are complex tori. A interesting class of homomorphism are isogenies. An *isogeny* between two complex tori T and T' is a surjective homomorphism $f : T \to T'$ with finite kernel. By definition we have that T and T' have the same dimension as complex tori.

Definition 1. *The degree deg of a homomorphism $f : T \to T'$ is the order of the group $ker(f)$, if it is finite, and 0 otherwise. Moreover the definition of the degree of a homomorphism extends to $Hom_{\mathbb{Q}}(T, T') := Hom(T, T') \otimes \mathbb{Q}$ by $deg(rf) = r^2 deg(f)$ for any $r \in \mathbb{Q}$ and $f \in Hom(T; T')$.*

Definition 2. *The exponent $e(f)$ of an isogeny f is the exponent of the finite group $ker(f)$, that is, the smallest positive integer n with $nx = 0$ for all $x \in ker(f)$.*

Proposition 3. *[3, Proposition 1.2.6] For any isogeny $f : T \to T'$ of exponent $e(f)$ there exists an isogeny $g : T' \to T$, unique up to isomorphisms, such that $gf = e(f)id_T$ and $fg = e(f)id_{T'}$.*

Let $T = V/\Lambda$ be a complex torus of dimension g. Then

$$V^* = \{g : V \to \mathbb{C} | \ \ g(\alpha v) = \overline{\alpha} g(v), g(v + v') = g(v) + g(v')\}$$

is a complex vector space of the same dimension as V . Define

$$\Lambda^* = \{g \in V^* | \ \ g(\Lambda) \subset \mathbb{Z}\}.$$

Note that Λ^* is a lattice in V^*, and $\widehat{T} := V^*/\Lambda^*$ is a complex torus of dimension g, called the *dual torus*. In addition, we have the natural identification $\widehat{\widehat{T}} = T$. Now, let $f : T \to T'$ be a homomorphism between complex torus. We define the *dual homomorphism* $\widehat{f} : \widehat{T'} \to \widehat{T}$ by setting $g \mapsto g \circ f$ for $g \in \widehat{T'}$.

Let L be a line bundle on T. For any point $x \in T$ the line bundle $t_x^* L \otimes L^{-1}$ has first Chern class zero. So, identifying $\widehat{T}$ with $Pic^0(T)$ (see [3, Proposition 2.4.1]) we get a homomorphism

$$\Phi_L : T \to \widehat{T}, \ x \mapsto t_x^* L \otimes L^{-1}.$$

Proposition 4. *[3, Corollary 2.4.6] Let L be a line bundle on T. Then* *(i)* Φ_L *depends only on the first Chern class of* L;

(ii) $\Phi_{L\otimes M} = \Phi_L + \Phi_M$ *for all* $L, M \in Pic(T)$;

(iii) $\widehat{\Phi_L} = \Phi_L$ *under the natural identification* $\widehat{\widehat{T}} = T$.

(iv) For any homomorphism $f : T \to T'$ *of complex tori the following diagram commmutes*

$$\begin{array}{ccc} T' & \xrightarrow{\Phi_L} & \widehat{T'} \\ f\uparrow & & \downarrow\widehat{f} \\ T & \xrightarrow{\Phi_{f^*L}} & \widehat{T} \end{array}$$

2.2 Abelian varieties

We are interested in complex tori admitting a polarization. But, what is a polarization on a complex torus $T = V/\Lambda$? First, recall that a *hermitian form* on V is a map $H : V \times V \to \mathbb{C}$, which is $\mathbb{C}$-linear in the first argument and satisfies $H(v, w) = \overline{H(w, v)}$ for all $v, w \in V$, and a *alternating form* on V is a map $E : V \times V \to \mathbb{R}$, which is $\mathbb{C}$-linear in each argument separately, such that $E(\Lambda, \Lambda) \subset \mathbb{Z}$ and $E(iv, iw) = E(v, w)$ for all $v, w \in V$. We say that H is a non degenerate form if $H(v, w) = 0$ for all w implies $v = 0$.

Proposition 5. *[3, Lemma 2.1.7] Let* V *be a complex vector space. There is a 1-1 correspondence between the set of hermitian forms* H *on* V *and the set of real valued alternating forms* E *on* V *satisfying* $E(iv, iw) = E(v, w)$, *given by* $E(v, w) = Im(H(v, w))$ *and* $H(v, w) = E(iv, w) + iE(v, w)$ *for all* $v, w \in V$.

Let A be a complex torus. A *polarization* on A is by definition the first Chern class $H = c_1(L)$ of a positive definite line bundle L on A. Or, equivalently, a polarization of a complex torus A is a positive definite hermitian form H on V which is non degenerate and satisfies $Im(H(\Lambda \times \Lambda)) \subseteq \mathbb{Z}$. By abuse of notation, we call our polarization L.

Definition 6. *An abelian variety is a complex torus* A *admitting a polarization. If* L *is a polarization on* A, *we call the pair* (A, L) *a polarized abelian variety.*

Let E be a alternating form. According to the Elementary Divisor Theorem (cf [8, Theorem 7.8]) there is a basis $\{\alpha_1, ..., \alpha_g, \beta_1, ...\beta_g\}$ of Λ, with respect to which E is given by the matrix

$$\begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix},$$

where $D = diag(d_1, ..., d_g)$, such that d_j are positive integers satisfying $d_j | d_{j+i}$, for $j = 1, ..., g-1$. The elementary divisors $d_1, ..., d_g$ given by the theorem are uniquely determined by E and Λ and thus by L. The vector $(d_1, ..., d_g)$ as well as the matrix D are called the *type* of the line bundle L, and the basis $\{\alpha_1, ..., \alpha_g, \beta_1, ...\beta_g\}$ is called a symplectic (or canonical) basis of Λ for L (or H or E).

Definition 7. *If (A, L) is a polarized abelian variety such that the type of L is $(1, ..., 1)$ then we say that (A, L) is a principally polarized abelian variety (p.p.a.v.). Moreover, if (A, L) is p.p.a.v., then Φ_L is an isomorphism and we identify A with its dual $\widehat{A}$.*

If L is of type $(d_1, ..., d_g)$, we define the degree of the polarization L to be the product $d_1 \cdot ... \cdot d_g$. Moreover, by Riemann-Roch's Theorem and [3, Proposition 5.2.3],

$$deg(L) = \frac{(L^g)}{g!}, \tag{2.1}$$

where (L^g) is the self-intersection number of a line bundle L on a g-dimensional complex torus. If we denote by $\chi(L) = \sum_{i=0}^{g}(-1)h^i(L)$ the *Euler-Poincaré characteristic* of L, by [3, Theorem 3.6.1 and Corollary 3.6.2] we have

$$\chi(L) = deg(L) \text{ and } \chi(L)^2 = deg(\Phi_L). \tag{2.2}$$

Fix a polarization L on A. Then L induces an isogeny $\Phi_L : A \to \widehat{A}$ depending only on the class of L in the Néron Severi group $NS(X)$. The exponent $e(L)$ of the finite group $K(L) = ker(\Phi_L)$ is called the exponent of the polarization L. By Proposition 3 there exists an unique isogeny $\Psi_L : \widehat{A} \to A$ such that $\Psi_L \Phi_L = e(L) id_A$ and $\Phi_L \Psi_L = e(L) id_{\widehat{A}}$. Thus Φ_L has an inverse in $Hom_{\mathbb{Q}}(\widehat{A}, A)$, namely

$$\Phi_L^{-1} = \frac{1}{e(L)} \Psi_L.$$

Let (A, L) be a polarized abelian variety of type $(d_1, ..., d_g)$. Then the dual abelian variety $\widehat{A}$ admits a polarization compatible with L.

Proposition 8. *[3, Proposition 14.4.1] Let (A, L) be a polarized abelian variety of type $(d_1, ..., d_g)$. There is a unique polarization L_δ on $\widehat{A}$ characterized by the following equivalent properties:*

(i) $\Phi_L^* L_\delta \equiv L^{d_1 1 d_g}$;

(ii) $\Phi_{L_\delta} \Phi_L = d_1 d_g id_A$.

The polarization L_δ *is of type* $(d_1, \frac{d_1 d_g}{d_{g-1}}, ..., \frac{d_1 d_g}{d_2}, d_g)$.

The polarization defined by L_δ is called the *dual polarization* and the pair $(\widehat{A}, L_\delta)$ is called the *dual polarized abelian variety.*

Now, let (A, L) be a polarized abelian variety and A' an abelian subvariety of A. Consider the canonical embedding $i : A' \hookrightarrow A$ and define the exponent of A' to be the exponent $e(A') := e(i^*L)$. Moreover, the norm-endomorphism of A associated to A' (with respect to L) is defined as

$$N_{A'} = i \circ \Psi_{i^*L} \circ \widehat{i} \circ \Phi_L,$$

that is, as the composition

$$N_{A'} : A \xrightarrow{\Phi_L} \hat{A} \xrightarrow{\hat{i}} \hat{A}' \xrightarrow{\Psi_{i^*L}} A' \xrightarrow{i} A. \tag{2.3}$$

Given a polarization L on A, we associate to every abelian subvariety A' of A a symmetric idempotent

$$\varepsilon_{A'} := \frac{1}{e(A')} N_{A'} \in End_{\mathbb{Q}}(A).$$

On the other hand, if ε is a symmetric idempotent in $End_{\mathbb{Q}}(A)$, there is an integer $n > 0$ such that $n\varepsilon \in End(A)$ and $A^\varepsilon = im(n\varepsilon)$ is independent of the choice of n. The assignments $\varphi : A' \mapsto \varepsilon_{A'}$ and $\psi : \varepsilon \mapsto A^\varepsilon$ are inverse to each other ([3, Theorem 5.3.2]).

Since the set of symmetric idempotents in $End_{\mathbb{Q}}(A)$ admits a canonical involution $\varepsilon \mapsto 1-\varepsilon$, by the above equivalence the polarization L of A induces a canonical involution on the set of abelian subvarieties of A given by $A' \mapsto B := A^{1-\varepsilon}$. The variety B is is called *complementary abelian subvariety* of A' in A (with respect to the polarization L).

Proposition 9. *[3, Corollary 12.1.5] Let* (A', B) *be a pair of complementary abelian subvarieties of a principally polarized abelian variety* (A, L) *with* $dimA' \geq dimB = r$. *If the induced polarization* i_B^*L *is of type* $(d_1, ..., d_r)$, *then* $i_{A'}^*L$ *is of type* $(1, ..., 1, d_1, ..., dr)$. *The integer* d_r *is the exponent of both* A' *and* B *as abelian subavarieties of* A.

2.3 Jacobian varieties

Let C be a smooth projective curve of genus $g = g(C)$ over the field of complex numbers. Recall that the genus of C is given by $g(C) = dimH^0(C, \Omega_C)$, where $H^0(C, \Omega_C)$ is the vector space of global holomorphic differential 1-forms on C. Moreover, the homology group $H_1(C, \mathbb{Z})$ is a free abelian group of rank $2g$. It is seen as a lattice in the dual space $H^0(C,\Omega_C)^* := Hom(H^0(C,\Omega_C),\mathbb{C})$ by identifying the closed cycles $\gamma \in H_1(C,\mathbb{Z})$ with the linear functions

$$\eta \in H^0(C,\Omega_C) \mapsto \int_\gamma \eta \in \mathbb{C}.$$

By [3, Lemma 11.1.1], the canonical map

$$H_1(C,\mathbb{Z}) \to H^0(C,\Omega_C)^*$$

is injective.

Definition 10. *The quotient*

$$J(C) := \frac{H^0(C,\Omega_C)^*}{H_1(C,\mathbb{Z})}$$

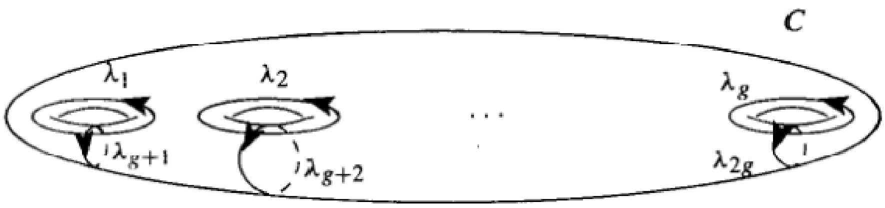
is a complex torus of dimension g, called the Jacobian variety of C

If $g = 0$, then $J(C) = 0$. Let $g \geq 1$. In order to describe $J(C)$ in terms of period matrices, choose bases $\lambda_1, ..., \lambda_{2g}$ of $H_1(C,\mathbb{Z})$ and $\eta_1, ..., \eta_g$ of $H^0(C,\Omega_C)$. Let $l_1, ..., l_g$ denote the basis of $H^0(C,\Omega_C)^*$ dual to $\eta_1, ..., \eta_g$, such that $l_i(\eta_j) = \delta_{ij}$ for $1 \leq i, j \leq g$. Considering λ_i as a linear form on $H^0(C,\Omega_C)$, as above, we have $\lambda_i = \sum_{j=1}^{g}(\int_{\lambda_i}\eta_j)l_j$ for $i = 1, ..., 2g$. Hence

$$\Pi = \begin{bmatrix} \int_{\lambda_1}\eta_1 & \cdots & \cdots & \int_{\lambda_{2g}}\eta_1 \\ . & & & . \\ . & & & . \\ . & & & . \\ \int_{\lambda_1}\eta_g & \cdots & \cdots & \int_{\lambda_{2g}}\eta_g \end{bmatrix}$$

is a period matrix for $J(C)$ with respect to these bases.

Now, fix a homology basis $\lambda_1, ..., \lambda_{2g}$ of $H_1(C,\mathbb{Z})$ with intersection matrix $\begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix}$ as indicated in the following picture.



Denote by E the alternating form on $H^0(C,\Omega_C)^*$ with matrix $\begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix}$ with respect to the basis $\lambda_1, ..., \lambda_{2g}$ and define

$$H : H^0(C, \Omega_C)^* \times H^0(C, \Omega_C)^* \to \mathbb{C} \;\; by \;\; H(u, v) = E(iu, v) + iE(u, v).$$

Proposition 11. *[3, Proposition 11.1.2] The hermitian form H defines a principal polarization on $J(C)$.*

The polarization H is called the canonical polarization of J(C). Any divisor Θ on $J(C)$ such that the line bundle $\mathcal{O}_{J(C)}(\Theta)$ defines the canonical polarization is called a theta divisor of the Jacobian $J(C)$. Then $(J(C), \Theta)$ is a principally polarized abelian variety of dimension $g(C)$.

By [3, Abel-Jacobi Theorem 11.1.3.] there is a canonical isomorphism between $Pic^0(C)$ and $J(C)$. Thus, consider $p \in C$ and define the map

$$\alpha_p : C \to J(C), \;\; p' \mapsto O_C(p - p').$$

Remark 12. *This map is called the Abel-Jacobi map and we can identify its dual map $\widehat{\alpha_p}$ with ${\Phi_\Theta}^{-1}$, via the isomorphism $\widehat{J(C)} \simeq J(C)$. Moreover, by [3, Corollary 11.1.5] α_p is an embedding.*

Theorem 13. *[3, Universal Property of the Jacobian 11.4.1] Let A be an abelian variety and $\varphi : C \to A$ be a rational map. Then there exists an unique homomorphism $\tilde{\varphi} : J(C) \to A$ such that for every $p \in C$ the following diagram is commutative*

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & A \\ \downarrow{\scriptstyle \alpha_p} & & \downarrow{\scriptstyle t_{-\varphi(p)}} \\ J(C) & \xrightarrow{\tilde{\varphi}} & A \end{array}$$

Corollary 14. *[3, Corollary 11.4.2.] Let $\varphi : C \to A$ be a rational map and Θ the polarization of $J(C)$. Then the dual of the homomorphism $\tilde{\varphi}$ is given by $\widehat{\tilde{\varphi}} = -\Phi_\Theta \circ \varphi^*$, via the isomorphism $\widehat{J(C)} \simeq J(C)$.*

Let $f : C \to C'$ be a finite morphism of smooth projective curves and $J(C)$ and $J(C')$ its Jacobian varieties, respectively. Let $p \in C$ and consider the composition $\alpha_{f(p)} \circ f : C \to J(C')$. By the previous theorem there is an unique homomorphism N_f, called the *norm map* of f, fitting into the following commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & C' \\ {\scriptstyle \alpha_p}\downarrow & & \downarrow{\scriptstyle \alpha_{f(p)}} \\ J(C) & \xrightarrow{N_f} & J(C') \end{array}$$

So, if Θ' is a theta divisor on $J(C')$, dualizing the equation $\alpha_{f(p)}f = N_f\alpha_p$ and applying Corollary 14 gives $\widehat{N_f}\Phi_{\Theta'} = \Phi_\Theta f^*$. The next result gives conditions for the pullback f^* to be an embedding.

Proposition 15. *[3, Proposition 11.4.3] The homomorphism $f^* : J(C') \to J(C)$ is not injective if and only if f factorizes via a cyclic étale covering f' of degree $n \geq 2$:*

$$\begin{array}{ccc} & & C'' \\ & \overset{f''}{\nearrow} & \downarrow{\scriptstyle f'} \\ C & \underset{f}{\searrow} & \\ & & C' \end{array}$$

Corollary 16. *[3, Proposition 11.4.4] For any finite morphism $f : C \to C'$ of smooth projective curves C and C' there is a factorization*

$$\begin{array}{ccc} & & C_e \\ & \overset{g}{\nearrow} & \downarrow{\scriptstyle f_e} \\ C & \underset{f}{\searrow} & \\ & & C' \end{array}$$

with f_e étale, $ker(f^*) = ker(f_e^*)$, and $g^* : J(C_e) \to J(C)$ injective.

2.4 Prym-Tyurin varieties and Prym varieties

We saw that to any smooth projective curve C one can associate a principally polarized abelian variety, its Jacobian. In this section we will study abelian subvarieties of the Jacobian $(J(C), \Theta)$ whose induced polarization is a multiple of a principal polarization.

Definition 17. *A principally polarized abelian variety (A, θ) is called a Prym-Tyurin variety of exponent e for C if A is an abelian subvariety of $J(C)$ and*

$$i^*\Theta \equiv e\theta,$$

where $i : A \hookrightarrow J(C)$ is the inclusion map. Note in this case that e is the exponent of A in $J(C)$.

Fix a point $p \in C$ and consider the Abel-Jacobi map $\alpha_p : C \to J(C)$. Let (A, θ) be a Prym-Tyurin variety for C and $i : A \hookrightarrow J(C)$ the inclusion map. Since θ is a principal polarization, we can identify A with its dual variety $\widehat{A}$ via the isomorphism Φ_θ and, considering the map (2.3), we can identify $\Psi_\theta \circ \widehat{i} \circ \Phi_\Theta$ with $\widehat{i}$. The composition

$$\varphi_A : C \xrightarrow{\alpha_p} J(C) \xrightarrow{\widehat{i}} A$$

is called the *Abel-Prym map* of A.

The following results characterize Prym-Tyurin varieties for C.

Proposition 18. *[3, Proposition 12.1.9] Consider (A', θ') and (A, θ) principally polarized abelian varieties. If $A' \subseteq A$ an abelian subvariety of exponent e, then the following statements are equivalent:*

(i) $ker(N_{A'})$ is connected,

(ii) $i^*\theta \equiv e\theta'$.

Theorem 19. *[3, Welters' Criterion 12.2.2] Let (A, θ) be a principally polarized abelian variety of dimension n and C a smooth projective curve. Then (A, θ) is a Prym-Tyurin variety of exponent e for the curve C if and only if there is a morphism $\varphi : C \to A$ such that*

(i) $\varphi^* : A \to J(C)$ is an embedding,

(ii) $\varphi_*[C] = \dfrac{e}{(n-1)!} \bigwedge^{n-1} \theta$ in $H^{2n-2}(A, \mathbb{Z})$.

An interesting question is when a principally polarized abelian variety coincides with $J(C)$. We have the following criterion due to Matsusaka.

Remark 20. *[3, Matsusaka's Criterion 12.2.5] Let (A, θ) be a principally polarized abelian variety of dimension n and $C \subset A$ an irreducible curve with $[C] = \dfrac{1}{(n-1)!} \bigwedge^{n-1} \theta$ in $H^{2n-2}(A, \mathbb{Z})$. Then C is smooth and $(A, \theta) \cong (J(C), \Theta)$, the Jacobian variety of C.*

Corollary 21. *[3, Corollary 12.2.6] For a principally polarized abelian variety (A, θ) and a smooth irreducible curve C the following conditions are equivalent:*

(i) (A, θ) is a Prym-Tyurin variety of exponent 1 for the curve C.

(ii) $(A, \theta) \cong (J(C), \Theta)$, the Jacobian variety of C.

Now we will find Prym-Tyurin varieties for a given curve C. Let $f : C \to C'$ be a morphism of degree k of smooth projective curves and assume that C' is non rational. We can associate in a natural way a subvariety A of the Jacobian $J(C)$.

Lemma 22. *[3, Lemma 12.3.1] Let $f : C \to C'$ be a morphism of degree k of smooth projective curves. Then $(f^*)^*\Theta = k\Theta'$ where Θ' is the polarization of $J(C')$.*

We want to associate to the covering f a Prym-Tyurin variety. If the complementary abelian variety A of $f^*J(C')$ in $J(C)$ is a Prym-Tyurin variety for C, we call it the *Prym variety* associated to the covering f. The next theorem says that there are exactly 3 types of coverings $f : C \to C'$ leading to Prym varieties: étale double coverings, double coverings ramified in 2 points, and genus 2 coverings of an elliptic curve.

Theorem 23. *[3, Theorem 12.3.3] Let $f : C \to C'$ be a finite morphism of degree $n \geq 2$ of smooth non rational projective curves. The the abelian subvariety A of $J(C)$, as defined above, is a Prym variety if and only if f is of one of the following types:*

(i) f is étale of degree 2.

(ii) f is of degree 2 and ramifed in 2 points

(iii) C is genus 2 and C' is of genus 1.

In cases (i) and (ii) the Prym variety A is of exponent 2. In (iii) consider the factorization $f = f_e \circ g$ of Corollary 16, where C_e is an elliptic curve, and we have $e(A) = deg(g)$.

The following result shows that when we are dealing with a Prym variety of types (i) or (ii), then the degree of the Abel-Prym map is known. Let $f : C \to C$ be a double covering of smooth projective curves of genus ≥ 1, étale or ramified in two points of C determining a Prym variety A. Consider $\varphi : C \to A$ its Abel-Prym map and let $\iota : C \to C$ be the involution corresponding to the double covering $f : C \to C'$.

Proposition 24. *[3, Proposition 12.5.2]*

(i) If C is not hyperelliptic, then $\varphi(p) = \varphi(q)$ for distinct points $p, q \in C$ if and only if f is ramified in p and q. In particular, φ is injective in the étale case.

(ii) If C is hyperelliptic, then $\varphi : C \to A$ is of degree 2 onto its image and $\varphi(p) = \varphi(q)$ for distinct points $p, q \in C$ if and only if $p + \iota(q)$ is in the unique linear system of degree 2 and dimension 1 on C.

CHAPTER 3

Generalized Abel-Prym maps

In this chapter, we present some results that will be used in the following chapters. In the first section, we consider (A, θ) a polarized abelian variety of dimension n with its polarization of degree $deg(\theta) = d$ and $\varphi : C \to A$ a morphism, where C is a smooth curve. We will prove results involving the map $\tilde{\varphi} \circ \tilde{\varphi}^t$ and the degree of the polarization θ. In a second moment, we consider generalized Abel-Prym maps, defined as the composition of the Abel map of a curve C with with the norm map of some abelian subvariety A of $J(C)$.

3.1 The transpose map

Definition 25. *Let $f : A \to A'$ be a morphism of abelian varieties and let θ and θ' be polarizations on A and A', respectively. We define the* transpose *of f as the morphism $A' \to A$ given by*

$$f^t = \Psi_\theta \circ \hat{f} \circ \Phi_{\theta'}.$$

where θ and θ' are polarizations of A and A', respectively.

By Proposition 4 the following diagram

$$\begin{array}{ccc} A' & \xrightarrow{\Phi_{\theta'}} & \widehat{A'} \\ {\scriptstyle f}\uparrow & & \downarrow{\scriptstyle \hat{f}} \\ A & \xrightarrow{\Phi_{f^*\theta'}} & \widehat{A} \end{array}$$

comutes, that is,

$$\widehat{f} \circ \Phi_{\theta'} \circ f = \Phi_{f^*\theta'}.$$

Thus

$$\Psi_\theta \circ \widehat{f} \circ \Phi_{\theta'} \circ f = \Psi_\theta \circ \Phi_{f^*\theta'},$$

and $f^t \circ f = \Psi_\theta \circ \Phi_{f^*\theta'}$ is an isogeny.

Let $\varphi : C \to A$ be a morphism. We have the morphism $\tilde{\varphi} : J(C) \to A$, and by the above definition $\tilde{\varphi}^t = \Psi_{\Theta_C} \circ \hat{\tilde{\varphi}} \circ \Phi_\theta$.

Proposition 26. *Let $\varphi^* : \hat{A} \to J(C)$ be the pullback map of φ. Then $\tilde{\varphi} \circ \tilde{\varphi}^t = \Phi_{(\varphi^*)^*\Theta_C} \circ \Phi_\theta$ and is an isogeny of degree $l^2 d^2$, where $l = \deg((\varphi^*)^*\Theta_C)$.*

Proof. Let Θ_δ be the dual polarization of Θ_C. Then Θ_δ is a polarization in $\widehat{J(C)}$ and $\Psi_{\Theta_C} = \Phi_{\Theta_C}^{-1} = \Phi_{\Theta_\delta}$. Moreover, by Proposition 4, we have $\tilde{\varphi} \circ \Phi_{\Theta_\delta} \circ \hat{\tilde{\varphi}} = \Phi_{\hat{\tilde{\varphi}}^*\Theta_\delta}$ and thus

$$\tilde{\varphi} \circ \tilde{\varphi}^t = \tilde{\varphi} \circ \Psi_{\Theta_C} \circ \hat{\tilde{\varphi}} \circ \Phi_\theta = \tilde{\varphi} \circ \Phi_{\Theta_\delta} \circ \hat{\tilde{\varphi}} \circ \Phi_\theta = \Phi_{\hat{\tilde{\varphi}}^*\Theta_\delta} \circ \Phi_\theta, \tag{3.1}$$

showing that $\tilde{\varphi} \circ \tilde{\varphi}^t$ is an isogeny.

To compute the degree we note that, by Corollay 14 and Proposition 8,

$$\hat{\tilde{\varphi}} = -\Phi_{\Theta_C} \circ \varphi^* \quad \text{and} \quad \Phi_{\Theta_C}^*\Theta_\delta = \Theta_C.$$

and consequently

$$\hat{\tilde{\varphi}}^*\Theta_\delta = (-\Phi_{\Theta_C} \circ \varphi^*)^*\Theta_\delta = (\varphi^*)^*\Phi_{\Theta_C}^*\Theta_\delta = (\varphi^*)^*\Theta_C. \tag{3.2}$$

Thus, by (2.2), (3.1) and (3.2)

$$\deg(\tilde{\varphi} \circ \tilde{\varphi}^t) = \deg(\Phi_{\hat{\tilde{\varphi}}^*\Theta_\delta}) \cdot \deg(\Phi_\theta) =$$

$$\chi(\hat{\tilde{\varphi}}^*\Theta_\delta)^2 \cdot \chi(\theta)^2 = \chi((\varphi^*)^*\Theta_C)^2 \cdot \chi(\theta)^2. =$$

$$\deg((\varphi^*)^*\Theta_C)^2 \cdot \deg(\theta)^2 = l^2 d^2$$

and we are done. □

To understand the next proposition, we need some definitions and results. Let A be an abelian variety of dimension n. An algebraic cycle V on A with coefficients in $\mathbb{Z}$ is a finite formal sum

$$V = \sum r_i V_i,$$

where r_i are integers and V_i algebraic subvarieties of A, which we assume to be all of the same dimension. If $dimV_i = p$, we call V an algebraic p-cycle. Let

$$V = \sum r_i V_i \text{ and } W = \sum s_j W_j$$

be algebraic p-cycles and q-cycles, respectively, on A of complementary dimension. We say that V and W intersect properly if $V_i \cap W_j$ is either of pure dimension $p + q - n = 0$ or empty, whenever $r_i \neq 0 \neq s_j$, for every i and j.

Let V and W be algebraic cycles on A of complementary dimension. Suppose V and W intersect properly, then the usual intersection product

$$V \cdot W = \sum u_i x_i$$

is a 0-cycle on A. The endomorphism $\delta(V, W)$ of A is given by

$$\delta(V, W)(x) = S(V \cdot (t_x^* W - W)),$$

where $S(V \cdot W) = u_1 x_1 + ... + u_n x_n \in A$. The following four results were taken from [3].

Proposition 27. *[3, Proposition 5.4.7] For any divisor D on A and $0 \leq r \leq n$ we have*

$$\delta\left(\bigwedge^{r} D, \bigwedge^{n-r} D\right) = \frac{n-r}{n}\left(\bigwedge^{n} D\right) id_A.$$

Proposition 28. *[3, Proposition 11.6.1] Let (A, θ) be a polarized abelian variety and let $\varphi : C \to A$ be a morphism, where C is a smooth curve. Then*

$$\delta(\varphi_*[C], \theta) = -\tilde{\varphi} \circ \tilde{\varphi}^t,$$

where $\tilde{\varphi} : J(C) \to A$ is induced by φ and $J(C)$ is considered with its polarization Θ.

Theorem 29. *[3, Theorem 4.11.1] Two algebraic cycles on an abelian variety are homologically equivalent if and only if they are numerically equivalent.*

Theorem 30. *[3, Theorem 11.6.4] Let L be a nondegenerate line bundle and Γ an algebraic 1-cycle on A. If $\delta(\Gamma, L) = 0$, then Γ is numerically equivalent to zero.*

Proposition 31. *Let (A, θ) be a polarized abelian variety of dimension n and let $\varphi : C \to A$ be a morphism, where C is a smooth curve. For any positive integer k, the following are equivalent:*

(i) $\tilde{\varphi} \circ \tilde{\varphi}^t = (kd)\, id_A$, *where* $d = \deg(\theta)$;

(ii) $\varphi_*[C] = \dfrac{k}{(n-1)!} \bigwedge^{n-1} \theta$.

Proof. By Proposition 27 we have

$$\delta\left(\frac{k}{(n-1)!}\bigwedge^{n-1}\theta, \theta\right) = \frac{k}{(n-1)!}\delta\left(\bigwedge^{n-1}\theta, \theta\right) = \left(\frac{k}{(n-1)!}\cdot\frac{-1}{n}\bigwedge^{n}\theta\right) id_A.$$

But, by (2.1), we have $\bigwedge^n \theta = \deg(\theta)n!$ and hence

$$\delta\left(\frac{k}{(n-1)!}\bigwedge^{n-1}\theta, \theta\right) = -\frac{kdn!}{n!} id_A = -(kd) id_A.$$

On the other hand, by Proposition 28 we have $\delta(\varphi_*[C], \theta) = -\tilde{\varphi} \circ \tilde{\varphi}^t$. Hence we have $\tilde{\varphi} \circ \tilde{\varphi}^t = (kd) id_A$ if and only if we have

$$\delta(\varphi_*[C], \theta) = \delta\left(\frac{k}{(n-1)!}\bigwedge^{n-1}\theta, \theta\right).$$

But, by Theorems 29 and 30, this happens if and only if

$$\varphi_*[C] = \frac{k}{(n-1)!}\bigwedge^{n-1}\theta.$$

□

Proposition 32. *Let* (A, θ) *be a polarized abelian variety of dimension* n *and let* $\varphi : C \to A$ *be a morphism, where* C *is a smooth curve. Let* $s \geq 1$ *be an integer and set* $r = d_1 d_n s$, *where* θ *is of type* $(d_1, \ldots, d_n)$. *The following are equivalent:*

(i) $\tilde{\varphi} \circ \tilde{\varphi}^t = r\, id_A$;

(ii) $(\varphi^*)^*\Theta_C \equiv s\, \theta_\delta$, *where* θ_δ *is the dual polarization of* θ.

Proof. ($\Rightarrow$) First, note that by Proposition 4 and Proposition 8

$$\Phi_{s\theta_\delta} \circ \Phi_{s\theta} = s^2(\Phi_{\theta_\delta} \circ \Phi_\theta) \quad \textit{and}$$

$$\Phi_{\theta_\delta} \circ \Phi_\theta = d_1 d_n id_A,$$

respectively. Thus, $\Phi_{s\theta_\delta} \circ \Phi_{s\theta} = s^2 \Phi_{\theta_\delta} \circ \Phi_\theta = s^2 d_1 d_n \, id_A = (sd_1)(sd_n) id_A$. Since the type of θ is $(d_1, ..., d_n)$, we have that the type of $s\theta$ is $(sd_1, ..., sd_n)$ and, again by Proposition 8, $s\theta_\delta$ is the dual polarization of $s\theta$.

Now, by Proposition 26, $\tilde{\varphi} \circ \tilde{\varphi}^t = \Phi_{(\varphi^*)^*\Theta_C} \circ \Phi_\theta$ and thus, by hypothesis we have $\Phi_{(\varphi^*)^*\Theta_C} \circ \Phi_\theta = r \, id_A = sd_1 d_n \, id_A$. Therefore,

$$\Phi_{(\varphi^*)^*\Theta_C} \circ \Phi_\theta = sd_1 d_n \, id_A$$

then

$$s\Phi_{(\varphi^*)^*\Theta_C} \circ \Phi_\theta = (sd_1)(sd_n) \, id_A$$

so

$$\Phi_{(\varphi^*)^*\Theta_C} \circ \Phi_{s\theta} = (sd_1)(sd_n) \, id_A$$

and by the uniqueness of the dual polarization,

$$(\varphi^*)^*\Theta_C \equiv s\theta_\delta.$$

($\Leftarrow$) Suppose $(\varphi^*)^*\Theta_C \equiv s\theta_\delta$. Then,

$$\tilde{\varphi} \circ \tilde{\varphi}^t = \Phi_{(\varphi^*)^*\Theta_C} \circ \Phi_\theta = \Phi_{s\theta_\delta} \circ \Phi_\theta = s(\Phi_{\theta_\delta} \circ \Phi_\theta) = sd_1 d_n \, id_A,$$

that is, $\tilde{\varphi} \circ \tilde{\varphi}^t = r \, id_A$.

□

As an immediate corollary of the previous results follows the Welters's criterion.

Corollary 33. *(Welters' criterion) Let (A, θ) be a polarized abelian variety of dimension n and let $\varphi : C \to A$ be a morphism, where C is a smooth curve. Let $(d_1 ..., d_n)$ be the type of θ, and set $d = d_1 \cdot \ldots \cdot d_n$ the degree of θ. The following are equivalent:*

(i) $\varphi_*[C] = \dfrac{k}{(n-1)!} \bigwedge^{n-1} \theta$;

(ii) $(\varphi^*)^*\Theta_C \equiv \dfrac{kd}{d_1 d_n} \theta_\delta$, *where θ_δ is the dual polarization of θ.*

Proof. Indeed, by Proposition 31

$$\varphi_*[C] = \frac{k}{(n-1)!} \bigwedge^{n-1} \theta \;\Leftrightarrow\; \tilde{\varphi} \circ \tilde{\varphi}^t = (kd) \, id_A$$

and taking $r = kd$ we have $s = \dfrac{kd}{d_1 d_n}$ and by the previous proposition

$$\tilde{\varphi} \circ \tilde{\varphi}^t = (kd)\, id_A \;\Leftrightarrow\; (\varphi^*)^* \Theta_C \equiv \frac{kd}{d_1 d_n} \theta_\delta.$$

□

3.2 Generalized Abel-Prym maps

Let C be a smooth non rational projective curve over the complex field $\mathbb{C}$ and let $(J(C), \Theta_C)$ be its principally polarized Jacobian. If A is an abelian subvariety of $J(C)$, not necessarily Prym-Tyurin, we define the generalized Abel-Prym map

$$\varphi : C \to A$$

to be the composition of the Abel map with the norm map of A. In addition, we define

$$C_A := \varphi(C). \tag{3.3}$$

Lemma 34. *C_A is a subcurve of A generating A. In particular, $g(C_A) \geq \dim(A)$.*

Proof. The first two assertions follow from the fact that $\alpha(C)$ is isomorphic to C and generates $J(C)$, and $\widehat{i}_A$ is surjective onto its image.

For the last assertion we consider the normalization map $\nu_A \colon C'_A \to C_A$ and the composition $f_A \colon C'_A \to A$ with the inclusion of C_A in A. Since the image of f_A generates A, then the map $\tilde{f}_A \colon J(C'_A) \to A$ induced by the universal property is surjective and thus $\dim(A) \leq g(C'_A) = g(C_A)$. □

By [7, Prop. II.6.8], the morphism

$$\varphi_A := C \to C_A \tag{3.4}$$

is finite and the curve C_A is complete, although possibly singular. It is not easy in this generality to determine when C_A is smooth. Moreover, the genus of C_A and the degree of φ_A seem to be rather difficult to compute. We will see some partial answers in the case where A is is one of the components of the isotypical decomposition of the Jacobian variety of C induced by a representation of a group G acting on C.

Let C be a curve on a polarized abelian variety (A, θ) with $dim A = n$ and $deg(\theta) = d$. The degree of C is defined as the intersection product $deg(C) := C \cdot \theta$ (cf [6]). Suppose C is algebraically equivalent to $\frac{k}{(n-1)!} \bigwedge^{n-1} \theta$, that is,

$$C \sim \frac{k}{(n-1)!} \bigwedge^{n-1} \theta.$$

Or equivalently,

$$i_*[C] = \frac{k}{(n-1)!} \bigwedge^{n-1} \theta,$$

with $i : C \hookrightarrow A$ the inclusion map. Then $deg(C) = (kd)n$ since $deg(\theta) = \dfrac{\theta^n}{n!}$. Now, if C is an irreducible curve that generates A, by [6, Proposition 4.1, p. 349],

$$k \geq \frac{\sqrt[n]{d}}{d}.$$

Proposition 35. *Let (A, θ) be a polarized abelian variety of dimension n and let $\varphi : C \to A$ be a morphism, where C is a smooth curve. Assume that $\varphi_*[C] = \dfrac{k}{(n-1)!} \bigwedge^{n-1} \theta$ and suppose that $\varphi(C)$ generates A. Then*

$$deg(\varphi) \leq \frac{kd}{\sqrt[n]{d}}.$$

where $d = \deg(\theta)$.

Proof. Let Γ be the normalization of $\varphi(C)$ and let $g : \Gamma \to A$ be the induced morphism. We have that $\varphi = g \circ h$, where $h : C \to \Gamma$. So,

$$\varphi_*[C] = g_*(h_*[C]) = g_*(deg(h)[h(C)]) = deg(\varphi) g_*[\Gamma].$$

Therefore,

$$g_*[\Gamma] = \frac{k}{deg(\varphi)(n-1)!} \bigwedge^{n-1} \theta,$$

that is,

$$i_*[\varphi(C)] = \frac{k}{deg(\varphi)(n-1)!} \bigwedge^{n-1} \theta,$$

where $i : \varphi(C) \hookrightarrow A$ the inclusion map. Thus, since $\varphi(C)$ generates A, by the previous argument, $deg(\varphi) \leq \dfrac{kd}{\sqrt[n]{d}}$.

□

Note that if A is a Prym-Tyurin variety for C of exponent $e(A) = e$, we have $deg(\varphi) \leq e$, since $deg(\theta) = 1$.

Proposition 36. *Let (A, θ) be a principally polarized abelian variety and let $\varphi : C \to A$ be a morphism, where C is a smooth curve. Assume that φ^* is an embedding and $\varphi_*[C] = \frac{k}{(n-1)!} \bigwedge^{n-1} \theta$. Then $\deg(\varphi) = k$ if and only if $\varphi(C)$ is smooth and $A = J(\varphi(C))$. In addition, if $\deg(\varphi) = 1$ then $A = J(C)$.*

Proof. Since

$$g_*[\Gamma] = \frac{k}{deg(\varphi)(n-1)!} \bigwedge^{n-1} \theta,$$

if $\deg(\varphi) = k$ we have that

$$g_*[\Gamma] = \frac{1}{(n-1)!} \bigwedge^{n-1} \theta$$

and consequently,

$$i_*[\varphi(C)] = \frac{1}{(n-1)!} \bigwedge^{n-1} \theta.$$

By Matsusaka's Criterion (Remark 20), $\varphi(C)$ is smooth and $A = J(\varphi(C))$. By Corollary 21 the converse is immediate.

□

The following result is a direct consequence of Lemma 22. For the sake of the completeness, we include it here, with a proof independent of this lemma.

Proposition 37. *Let $f\colon C \to C'$ be a finite morphism of smooth curves and assume that the pullback $f^*\colon J(C') \to J(C)$ is an embedding. Then $J(C')$ is a Prym-Tyurin variety of exponent $\deg(f)$ for C.*

Proof. We'll apply Welters' criterion (Theorem 19). Consider the composition $h := \alpha_{C'} \circ f\colon C \to J(C')$, where $\alpha_{C'}$ is the Abel map of C'. Then $h^* = f^* \circ \alpha_{C'}^*$ is an embedding, since $\alpha_{C'}^*$ is an isomorphism. Moreover, if $\deg(f) = q$ then $f_*[C] = q[C']$. This completes the proof, since by Poincaré's formula [3, Proposition 11.2.1] we have

$$[C'] = \frac{q}{(g-1)!} \bigwedge^{g-1} \Theta_{C'},$$

where $\Theta_{C'}$ is the canonical theta divisor of $J(C')$ and g is the genus of B. □

We remark that the hypothesis of f^* being an embedding is fundamental in Proposition 37, as shown in the following example. Consider $f : C \to C'$ a non-ramified morphism of degree 2 between a curve C of genus 3 and C' of genus 2. By Theorem 23, the complementary subvariety A of $f^*(J(C'))$ in $J(C)$ is a (classical) Prym variety, hence A is a Prym-Tyurin variety of exponent 2. Therefore the polarization of $f^*(J(C'))$ is of type $(1, 2)$, by Corollary 9, and it is not a Prym-Tyurin variety with respect to C. Note that, in this case, the exponent of $f^*(J(C'))$ as an abelian subvariety of $J(C)$ is 2, hence equal to the degree of f.

By Proposition 15, the pullback map f^* is injective if and only if f does not factor via a cyclic étale covering of degree ≥ 2. This implies that, in the case of an étale covering $f\colon C \to C'$, then $J(C')$ is not a Prym-Tyurin variety for C. The next result shows that even if we consider the image $f^*(J(C'))$ of the pullback map, wich is already an abelian subvariety of $J(C)$, then we still do not have a Prym-Tyurin variety for C.

Proposition 38. *Let $f\colon C \to C'$ be a non-constant cyclic étale morphism of smooth curves, where $g(C') \geq 2$ and consider the inclusion map $i\colon f^*(J(C')) \hookrightarrow J(C)$. If $\deg(f) \neq a^{g(C')}$ for some $a \in \mathbb{Z}$, then the induced polarization $i^*\Theta_C$ on $f^*(J(C'))$ is not a multiple of a principal polarization.*

Proof. Set $i : f^*(J(C')) \hookrightarrow J(C)$ to be the inclusion map, so f^* factors as $f^* = i \circ j$, where $j\colon J(C') \to f^*(J(C'))$ is an isogeny. Denote by Θ and Θ' the theta divisors on $J(C)$ and $J(C')$, respectivelly.

Set $n = \deg(f)$, by [3, Lemma 12.3.1], we have $(f^*)^*\Theta \equiv n\Theta'$. Hence the type of $(f^*)^*\Theta$ is $(n, \dots, n)$ and by [3, Theorem 3.6.1], we have

$$\chi((f^*)^*\Theta) = (-1)^s n^{g'}$$

for some $s \in \mathbb{Z}$. Now, $(f^*)^* = j^* \circ i^*$ and by [3, Corollary 3.6.6] we have

$$\chi((f^*)^*\Theta) = \chi(j^*(i^*\Theta)) = \deg(j)\chi(i^*\Theta).$$

Since f is étale of degree n, then the isogeny j also has degree n and we get that

$$\chi(i^*\Theta) = (-1)^s n^{g'-1}. \tag{3.5}$$

If $i^*\Theta$ was a multiple of a principal polarization on $f^*(J(C'))$, then it would be of type $(m, \dots, m)$ for some $m \in \mathbb{Z}$ and again by [3, Theorem 3.6.1], we would have

$\chi(i^*\Theta) = (-1)^r m^{g'}$ for some $r \in \mathbb{Z}$. Thus, (3.5) gives us

$$m^{g'} = n^{g'-1}.$$

We need to show this can only happen when $n = a^{g'}$ for some $a \in \mathbb{Z}$. Let $n = p_1^{r_1} \dots p_k^{r_k}$ be the prime decomposition of n. Then

$$n^{g'-1} = p_1^{r_1(g'-1)} \dots p_k^{r_k(g'-1)}$$

and for this to be of the form $m^{g'}$ we most have

$$r_i(g'-1) = s_i g'$$

for some $s_i \in \mathbb{Z}$, for every $i = 1, \dots, k$. Since g' and $g'-1$ are coprimes, then g' divides r_i and we may write $r_i = t_i g'$ for some $t_i \in \mathbb{Z}$, for $i = 1, \dots, k$. But then

$$n = (p_1^{t_1} \dots p_k^{t_k})^{g'}$$

and the result is proven. □

CHAPTER 4

The Group Algebra Decomposition of an Jacobian variety

In this chapter we will prove some results that allow us to study the degree of generalized Abel-Prym maps, when the subvarieties A_i of the Jacobian variety $J(C)$ are the components of its isotypical decomposition. For this, in the first section, we define such decompositions for an abelian varieties, not necessarily a Jacobian variety. In the following section, we apply this definition to the particular case of Jacobian varieties and prove some interesting results about these subvarieties.

4.1 Group actions on abelian varieties

First, remember that given a finite group G and a complex vector space V, a *representation* of V in G is a homomorphism $\rho : G \to GL(V)$, where to each element $g \in G$ we associate an element $\rho_g \in GL(V)$ which is a linear invertible operator. A subspace W of V is G-stable (or invariant) if $g \cdot w \in W$ for all $g \in G$ and $w \in W$. We say that ρ is an *irreducible representation* if it has no nontrivial invariant subspaces. The *character* of the representation ρ is the map $\chi : G \to \mathbb{C}$ given by $\chi(g) := Tr(\rho(g))$.

Let A be an abelian variety and let G denote a finite group acting on A. The map $\xi_g : A \to A$, given by the group action $x \mapsto g \cdot x$ for all $g \in G$ induces an algebra homomorphism

$$\rho : \mathbb{Q}[G] \to End_{\mathbb{Q}}(A),$$

where $\mathbb{Q}[G]$ denotes the rational group algebra of G. We saw in Chapter 2 that if $\alpha \in \mathbb{Q}[G]$, identifying α and $\rho(\alpha)$, there is an integer $n > 0$ such that $n\alpha \in End(A)$, so that $A^{\alpha} = im(n\alpha) \subset A$. However, to obtain proper abelian subvarieties we have to choose suitable elements α of $\mathbb{Q}[G]$. Since $\mathbb{Q}[G]$ is a semi-simple $\mathbb{Q}$-algebra (a direct sum of simple modules), we have

$$\mathbb{Q}[G] = Q_1 \times ... \times Q_r,$$

where Q_i are simple $\mathbb{Q}$-algebras. Consider the decomposition of the unit element $1 = e_1 + ... + e_r$. The elements $e_i \in Q_i$, seen as elements of $\mathbb{Q}[G]$, form a set of orthogonal idempotents contained in the center of $\mathbb{Q}[G]$. The following result guarantees a decomposition for the abelian variety.

Proposition 39. *[3, Proposition 13.6.1] Let $A_i = A^{e_i}$ for $i = 1, ..., r$.*

(i) A_i is a G-stable abelian subvariety of A with $Hom_G(A_i, A_j) = 0$ for $i \neq j$.

(ii) The addition map induces an isogeny

$$A_1 \times ... \times A_r \to A.$$

This decomposition is called the *group algebra or isotypical decomposition* of A. The subvarieties A_i are uniquely determined as images of central symmetric idempotents $e_i \in Q[G]$, which can be obtained as follows: Let χ_i be a character of an irreducible representation of $\rho_i\colon G \to GL(V_i)$, where V_i is a complex vector space corresponding to A_i, and let L_i be the field

$$L_i := \mathbb{Q}(\chi_i(g), g \in G),$$

then

$$e_i := \frac{\deg \chi_i}{|G|} \sum_{g \in G} \mathrm{tr}_{L_i|\mathbb{Q}}(\chi_i(g^{-1}))g \in \mathbb{Q}[G].$$

4.2 The case of Jacobian varieties

Let G be a group acting on a smooth curve C. Consider the isotypical decomposition of the Jacobian $J(C)$

$$J(C) \to A_1 \times ... \times A_n$$

induced by the action of G on $J(C)$, defined in the previous section.

For each $i = 1, ..., n$, let K_i be the kernel of ρ_i and consider the quotient curve with induced morphism

$$\psi_i \colon C \to \tilde{C}_i = C/K_i. \tag{4.1}$$

Note that $\tilde{C}_i$ is smooth, ψ_i has degree $|K_i|$ and, if the action of G on C is known, it is easy to compute the genus of $\tilde{C}_i$.

Lemma 40. *With the above notation, A_i is an abelian subvariety of $\psi_i^*(J(\tilde{C}_i))$.*

Proof. By [5, Prop. 5.2] we have $\psi_i^*(J(\tilde{C}_i)) = Im(p_{K_i})$, where

$$p_{K_i} = \frac{1}{|K_i|}\sum_{k\in K_i} k \ \in \mathbb{Q}[G].$$

Now, we have $\chi_i(g^{-1}) = \chi_i(kg^{-1}k^{-1}) = \chi_i(g^{-1}k^{-1})$, for all $k \in K_i, g \in G$, where the first equality follows from the property of the character and the second one from the fact that $K_i = ker(\rho_i)$. So,

$$\begin{aligned} p_{K_i}e_i &= \left(\frac{1}{|K_i|}\sum_{k\in K_i} k\right)\left(\frac{\deg\chi_i}{|G|}\sum_{g\in G}\operatorname{tr}_{L_i|\mathbb{Q}}(\chi_i(g^{-1}))g\right) \\ &= \frac{\deg\chi_i}{|G||K_i|}\sum_{k\in K_i}\sum_{g\in G}\operatorname{tr}_{L_i|\mathbb{Q}}(\chi_i(kg^{-1}k^{-1}))kg \\ &= \frac{\deg\chi_i}{|G||K_i|}\sum_{g\in G}\sum_{k\in K_i}\operatorname{tr}_{L_i|\mathbb{Q}}(\chi_i(g^{-1}k^{-1}))kg, \end{aligned}$$

and setting $h = kg$, we have

$$\begin{aligned} p_{K_i}e_i &= \frac{\deg\chi_i}{|G||K_i|}\sum_{g\in G}\sum_{h\in K_i g}\operatorname{tr}_{L_i|\mathbb{Q}}(\chi_i(h^{-1}))h, \\ &= \frac{|K_i|\deg\chi_i}{|G||K_i|}\sum_{h\in G}\operatorname{tr}_{L_i|\mathbb{Q}}(\chi_i(h^{-1}))h. \end{aligned}$$

Thus $p_{K_i}e_i = e_i$ and $A_i = Im(e_i) \subset Im(p_i) = \psi_i^*(J(\tilde{C}_i))$. □

With the notation of (3.3) and (3.4), we set $\varphi_i = \varphi_{A_i}$ and $C_i = C_{A_i}$, so that we have

$$\varphi_i \colon C \to C_i. \tag{4.2}$$

Proposition 41. *There is a morphism $f_i\colon \tilde{C}_i \to C_i$ such that $\varphi_i = f_i \circ \psi_i$.*

Proof. First note that given an action of G on a complex vector space V, we have a representation of $\rho\colon G \to GL(V)$. If K is the kernel of this representation, since $K = \bigcap G_x$, where for each $x \in V$ we have $G_x = \{g \in G| \;\; g \cdot x = x\}$. Then we have an isomorphism $V \to V/K$, given by $x \mapsto O_x = \{y = g \cdot x| \;\; g \in G\}$. In our case, we have a representation $\rho_i\colon G \to GL(V_i)$ with $K_i = ker(\rho_i)$. The trivial action of K_i on V_i induces a trivial action of K_i on A_i. Thus, we have compatible actions of K_i on C and A_i and then we can extend the morphism $\varphi_i : C \to A_i$ to $\overline{\varphi_i} : C/K_i \to A_i/K_i$. In addition, as $C/K_i = \tilde{C}_i$, we have $A_i/K_i \cong A_i$ and $Im\overline{\varphi_i} = C_i$, and so there is a morphism $f_i\colon \tilde{C}_i \to C_i$.

□

Proposition 42. *Let $A \subset A'$ be abelian subvarieties of the Jacobian of a curve C. Assume that (A', θ') and (A, θ) are Prym-Tyurin varieties of exponents e' and e, respectively, for C. Then*

$$e = e' \cdot e_{A'}(A),$$

where $e_{A'}(A)$ is the exponent of A as a subvariety of A'.

Proof. Denote by $i\colon A' \hookrightarrow J(C)$, $j\colon A \hookrightarrow J(C)$ and $h\colon A \hookrightarrow A'$ the inclusion maps such that $j = i \circ h$. By hypothesis , we have

$$i^*\Theta = e'\,\theta'$$

and

$$j^*\Theta = e\,\theta.$$

Moreover,

$$e\theta = j^*\Theta = h^*(i^*\Theta) = h^*(e'\theta') = e'h^*\theta',$$

which implies that,

$$h^*\theta' = \frac{e}{e'}\theta.$$

Now, writing $e_{A'}(A) = e(h^*\theta)$ and $k = \frac{e}{e'}$, by [9, Lemma 6.2], we have,

$$e_{A'}(A) = e(h^*\theta) = e(k\theta) = ke(\theta) = k.$$

Therefore, $e_{A'}(A) = \dfrac{e}{e'}$.

□

Theorem 43. *Assume A_i is a Prym-Tyurin variety for C. We have:*

(i) $\deg(f_i) \leq \dfrac{e(A_i)}{|K_i|}$. *In particular, if* $e(A_i) = |K_i|$ *then* f_i *is a normalization.*

(ii) If ψ_i^* *is an embedding then* A_i *is a Prym-Tyurin variety of exponent* $\dfrac{e(A_i)}{|K_i|}$ *for* $\tilde{C}_i$. *In particular,* $e(A_i) = |K_i|$ *if and only if* $A_i = \psi_i^*(J(\tilde{C}_i))$.

Proof. (i) Since A_i is Prym-Tyurin for C, $\deg(\varphi_i) \leq e(A_i)$, by Proposition 35. As $\deg(\psi_i) = |K_i|$ and $\deg(\varphi_i) = \deg(f_i)\deg(\psi_i)$, we have $\deg(f_i) \leq \dfrac{e(A_i)}{|K_i|}$. Moreover, if $e(A_i) = |K_i|$ then $deg(f_i) \leq 1$ and thus $deg(f_i) = 1$.

(ii) For the second statement, we assume ψ_i^* is an embedding. Then $J(\tilde{C}_i)$ is a Prym-Tyurin variety for C of exponent $|K_i|$, by Proposition 37. Denote by $i\colon J(\tilde{C}_i) \to J(C)$ the inclusion. Then $i^*\Theta_C = |K_i|\Theta_{\tilde{C}_i}$. On the other hand, since A_i is a Prym-Tyurin variety of exponent $e(A_i)$ for C, there is a principal polarization θ of A_i such that $j^*\Theta_C = e(A_i)\theta$, where $j\colon A_i \to J(C)$ is the inclusion. Now, by Lemma 40, A_i is an abelian subvariety of $J(\tilde{C}_i)$ and we let $h\colon A_i \to J(\tilde{C}_i)$ be the inclusion. Then $j = i \circ h$ and thus

$$e(A_i)\theta = j^*\Theta_C = h^*(i^*\Theta_C) = |K_i|h^*\Theta_{\tilde{C}_i}.$$

Hence $h^*\Theta_{\tilde{C}_i} = \dfrac{e(A_i)}{|K_i|}\theta$, thus showing that A_i is a Prym-Tyurin variety of exponent $\dfrac{e(A_i)}{|K_i|}$ for $\tilde{C}_i$. Finally, if $e(A_i) = |K_i|$, then A_i is a Prym-Tyurin variety of exponent 1 for $\tilde{C}_i$ and hence, by Corollary 21, we must have $A_i \cong J(\tilde{C}_i)$, which implies $A_i = \psi_i^*(J(\tilde{C}_i))$.

□

CHAPTER 5

Examples

In this chapter, we will apply the results of the previous chapters to calculate the degrees of some generalized Abel-Prym maps related to isotypical decomposition of Jacobians varieties.

5.1 Action of S_3 over a curve of genus 3

Consider $S_3 = \langle a, b | \quad a^3 = b^2 = 1, bab = a^{-1} \rangle$ the symmetric group of order 6 acting on a curve C of genus 3. Let

$$J(C) = \frac{\langle \alpha_i, \beta_i \rangle_{\mathbb{C}}}{\langle \alpha_i, \beta_i \rangle_{\mathbb{Z}}}, \quad i = 1, ..., 3$$

be the Jacobian variety of C written on the symplectic basis. The action of S_3 on $J(C)$ induced by the action on C is given by

$$\begin{array}{llll} a(\alpha_1) = \alpha_3, & a(\alpha_2) = \alpha_1 & \text{and} & a(\alpha_3) = \alpha_2; \\ a(\beta_1) = \beta_3, & a(\beta_2) = \beta_1 & \text{and} & a(\beta_3) = \beta_2; \\ b(\alpha_1) = -\alpha_2, & b(\alpha_2) = -\alpha_1 & \text{and} & b(\alpha_3) = -\alpha_3; \\ b(\beta_1) = -\beta_2, & b(\beta_2) = -\beta_1 & \text{and} & b(\beta_3) = -\beta_3. \end{array}$$

Let us present the isotypical decomposition of $J(C)$. The character table of S_3 associated with irreducible representations ρ_0, ρ_1 and ρ_2 (see [14, pag. 225]) is given by

	id	a, a^2	b, ab, a^2b
χ_0	1	1	1
χ_1	1	1	-1
χ_2	2	-1	0

For each $i = 0, 1, 2$, the central symmetric idempotent is given by

$$e_i = \frac{\deg \chi_i}{6} \sum_{g \in S_3} \overline{\chi_i(g)} g,$$

so,

$$e_0 = \frac{1}{6}(1 + a + a^2 + b + ab + a^2b),$$
$$e_1 = \frac{1}{6}(1 + a + a^2 - b - ab - a^2b),$$
$$e_2 = \frac{2}{6}(2 - a - a^2).$$

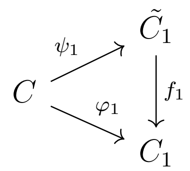
Since A_i are uniquely determined as images of central symmetric idempotents e_i, we have that $J(C) \simeq A_0 \times A_1 \times A_2$. The abelian subvariety $A_0 = J(C/S_3)$ is a trivial variety. The abelian subvariety $A_1 \subset J(C)$ is an abelian variety of dimension 1 (an elliptic curve) and type (3), given by

$$\frac{\langle 3\epsilon, 3\delta\rangle_{\mathbb{C}}}{\langle 3\epsilon, 3\delta\rangle_{\mathbb{Z}}},$$

with

$$\epsilon = \frac{\alpha_1 + \alpha_2 + \alpha3}{3} \quad \text{and} \quad \delta = \frac{\beta_1 + \beta_2 + \beta_3}{3}.$$

Since $\rho_1 : S_3 \to \mathbb{C}$ is the sign representation, we have that $K_1 = \langle a \rangle$ and $|K_1| = 3$, that is, ψ_1 has degree 3. By Proposition 41 there is a morphism f_1 such that the following diagram



is commutative, where $\tilde{C}_1 = C/K_1$. Moreover, since A_1 is Prym-Tyurin for C with $e(A_1) = 3$, by Proposition 9, and $|K_1| = 3$, we have by Theorem 43, that f_1 is a normalization.

Thus, $deg(\varphi_1) = def(\psi_1) = 3$, too. By Corollary 16, ψ_1^* is an embedding. Since $\tilde{C}_1$ is smooth and $g(C) = 3$, we have that $g(\tilde{C}_1) \leq 3$. If $g(\tilde{C}_1) = 2$ or $g(\tilde{C}_1) = 3$, by Riemann-Hurwitz formula applyed to the map ψ_1,

$$2g(C) - 2 = deg(\psi_1)(2g(\tilde{C}_1) - 2) + R$$

$$R = 4 - 3(2g(\tilde{C}_1) - 2)$$

$$R = -2 \quad \text{or} \quad R = -8,$$

and we have $R < 0$. Thus, $g(\tilde{C}_1) = 1$ and $A_1 = \psi_1^*(J(\tilde{C}_1))$. Moreover, as f_1 is a normalization, $g(C_1) = 1$.

On the other hand, the abelian variety $A_2 \subset J(C)$ given by

$$\frac{\langle \epsilon_1 - \epsilon_2, \epsilon_1 + 2\epsilon_2, \delta_1 - \delta_2, \delta_1 + 2\delta_2 \rangle_{\mathbb{C}}}{\langle \epsilon_1 - \epsilon_2, \epsilon_1 + 2\epsilon_2, \delta_1 - \delta_2, \delta_1 + 2\delta_2 \rangle_{\mathbb{Z}}},$$

with

$$\epsilon_1 = 2\alpha_1 - \alpha_2 - \alpha_3 \quad \text{and} \quad \epsilon_2 = -\alpha_1 + 2\alpha_2 - \alpha_3;$$

$$\delta_1 = 2\beta_1 - \beta_2 - \beta_3 \quad \text{and} \quad \delta_2 = -\beta_1 + 2\beta_2 - \beta_3,$$

is a variety of dimension 2 and type (1 3) that is not Prym-Tyurin for C. The 2-dimensional representation $\rho_2 : S_3 \to GL(\mathbb{C}^2)$ has kernel $K_2 = \{1\}$, so $|K_2| = 1$ and $\psi_2 : C \to \tilde{C}_2 = C$ has degree 1. Consequently $\varphi_2 = f_2$. In this case, we know nothing about the degree of the map φ_2.

5.2 Action of $\mathbb{Z}_2$ over a curve of genus 3

Consider $\mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$ the cyclic group of order 2 acting on a curve C of genus 3. Let

$$J(C) = \frac{\langle \alpha_i, \beta_i \rangle_{\mathbb{C}}}{\langle \alpha_i, \beta_i \rangle_{\mathbb{Z}}}, \quad i = 1, ..., 3$$

be the Jacobian variety of C written on the symplectic basis. The action of $\mathbb{Z}_2$ on $J(C)$ induced by the action on C is given by

$$\overline{1}(\alpha_1) = \alpha_3, \quad \overline{1}(\alpha_2) = \alpha_2 \quad \text{and} \quad \overline{1}(\alpha_3) = \alpha_1;$$
$$\overline{1}(\beta_1) = \beta_3, \quad \overline{1}(\beta_2) = \beta_2 \quad \text{and} \quad \overline{1}(\beta_3) = \beta_1;$$

Let us present the isotypical decomposition of $J(C)$. The character table of $\mathbb{Z}_2$

associated the trivial representation ρ_0 and the sign representation ρ_1, is given by

	$\overline{0}$	$\overline{1}$
χ_0	1	1
χ_1	1	-1

In our case, for each $i = 0, 1$, the central symmetric idempotent is given by

$$e_i = \frac{\deg \chi_i}{2} \sum_{g \in \mathbb{Z}_2} \overline{\chi_i(g)} g.$$

So,

$$e_0 = \frac{1}{2}(\overline{0} + \overline{1}) \quad \text{and} \quad e_1 = \frac{1}{2}(\overline{0} - \overline{1}).$$

Since A_i are uniquely determined as images of central symmetric idempotents e_i, we have that $J(C) \simeq A_0 \times A_1$. The abelian variety $A_0 = J(C/\mathbb{Z}_2)$ given by

$$\frac{\langle 2\epsilon_1, 2\epsilon_2, 2\delta_1, 2\delta_2 \rangle_{\mathbb{C}}}{\langle 2\epsilon_1, 2\epsilon_2, 2\delta_1, 2\delta_2 \rangle_{\mathbb{Z}}}$$

with

$$\epsilon_1 = \frac{\alpha_1 + \alpha_3}{2}, \quad \text{and} \quad \epsilon_2 = \alpha_2;$$

$$\delta_1 = \frac{\beta_1 + \beta_3}{2} \quad \text{and} \quad \delta_2 = \beta_2,$$

is a 2-dimensional variety of type (1 2) and hence it is not Prym-Tyurin for C. Let ρ_0 be the trivial representation and $K_0 = \mathbb{Z}_2$ its kernel. So, $\psi_0 : C \to \tilde{C}_0 = C/\mathbb{Z}_2$ has degree $|K_0| = 2$ and, since $g(\tilde{C}_0) = 2$, we have $A_0 = \psi_0^*(J(\tilde{C}_0))$. As A_0 is not a Prym-Tyurin variety our information on the degree of φ_0 is reduced to

$$deg(\varphi_0) = 2deg(f_0) \geq 2.$$

Now, $A_1 \subset J(C)$ is a 1- dimensional variety of type (2), given by

$$\frac{\langle 2\epsilon, 2\delta \rangle_{\mathbb{C}}}{\langle 2\epsilon, 2\delta \rangle_{\mathbb{Z}}}$$

with

$$\epsilon = \frac{\alpha_1 - \alpha_3}{2} \quad \text{and} \quad \delta = \frac{\beta_1 - \beta_3}{2}.$$

Note that A_1 is a Prym-Tyurin variety for C of exponent $e(A_1) = 2$ and as ρ_1 is the sign representation with kernel $K_1 = \{\overline{0}\}$, we have $deg(\psi_1) = 1$. Moreover, $deg(\varphi_1) \leq 2$, by Proposition 35. By Proposition 36, $deg(\varphi_1) = 2$ if and only if C_1 is smooth and $A_1 = J(C_1)$. Since $dim(A_1) = 1$ and $C_1 \subset A_1$ we have that $C_1 = A_1$ is a smooth curve of genus 1 and thus, $A_1 = J(C_1) = C_1$. Again, by Proposition 36, we must have

$$deg(\varphi_4) = 2.$$

In the two previous examples we had some subvarieties that were not Prym-Tyurin, and in these cases we find it difficult to obtain information about the degree of the map φ_i. In addition, the Prym-Tyurin varieties were all 1-dimensional. Now, we will see an example of decomposition where all subvarieties are Prym-Tyurin, including a 2-dimensional subvariety.

5.3 Action of D_4 over a curve of genus 4

Consider $D_4 = \langle a, b; a^4 = b^2 = (ab)^2 = 1\rangle$ the dihedral group of order 8 acting on a curve C of genus 4. Let

$$J(C) = \frac{\langle \alpha_i, \beta_i\rangle_{\mathbb{C}}}{\langle \alpha_i, \beta_i\rangle_{\mathbb{Z}}}, \quad i = 1, ..., 4$$

be the Jacobian variety of C written on the symplectic basis. The action of D_4 on $J(C)$ induced by action of D_4 on C is given by

$$\begin{array}{lllll} a(\alpha_1) = \alpha_2, & a(\alpha_2) = \alpha_3, & a(\alpha_3) = \alpha_4 & \text{and} & a(\alpha_4) = \alpha_1; \\ a(\beta_1) = \beta_2, & a(\beta_2) = \beta_3, & a(\beta_3) = \beta_4 & \text{and} & a(\beta_4) = \beta_1; \\ b(\alpha_1) = -\alpha_2, & b(\alpha_2) = -\alpha_1, & b(\alpha_3) = -\alpha_4 & \text{and} & b(\alpha_4) = -\alpha_3; \\ b(\beta_1) = -\beta_2, & b(\beta_2) = -\beta_1, & b(\beta_3) = -\beta_4 & \text{and} & b(\beta_4) = -\beta_3. \end{array}$$

Let us present the isotypical decomposition of $J(C)$. There are four irreducible representations of degree one and just one of degree two. The character table of D_4 associated with irreducible representations ρ_0,..., ρ_4 (see [14]) is given by

	1	$\{a^2\}$	$\{a, a^3\}$	$\{b, a^2b\}$	$\{ab, a^3b\}$
χ_0	1	1	1	1	1
χ_1	1	1	1	-1	-1
χ_2	1	1	-1	1	-1
χ_3	1	1	-1	-1	1
χ_4	2	-2	0	0	0

Again, for each $i = 0, 1, 2, 3$, the central symmetric idempotent is given by

$$e_i = \frac{\deg \chi_i}{8} \sum_{g \in \mathbb{Z}_2} \overline{\chi_i(g)} g.$$

Thus,

$$e_0 = \frac{1}{8}(1 + a + a^2 + a^3 + b + ab + a^2b + a^3b),$$

$$e_1 = \frac{1}{8}(1 + a + a^2 + a^3 - b - ab - a^2b - a^3b),$$

$$e_2 = \frac{1}{8}(1 - a + a^2 - a^3 + b - ab + a^2b - a^3b),$$

$$e_3 = \frac{1}{8}(1 - a + a^2 - a^3 - b + ab - a^2b + a^3b)$$

and

$$e_4 = \frac{1}{4}(1 - 2a^2).$$

From the idempotent elements we have $J(C) \simeq A_0 \times A_1 \times A_2 \times A_3 \times A_4$, where A_0 and A_3 are trivial abelian varieties.

The abelian varieties A_1 and A_2 have similar characteristics (the same type, exponent and dimension). Therefore, we will consider only A_1. We have that A_1 is given by

$$\frac{\langle 4\epsilon, 4\delta \rangle_{\mathbb{C}}}{\langle 4\epsilon, 4\delta \rangle_{\mathbb{Z}}}$$

with

$$\epsilon = \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{4} \quad \text{and} \delta = \frac{\beta_1 + \beta_2 + \beta_3 + \beta_4}{4}.$$

It is a 1-dimensional variety of type (4), Prym-Tyurin for C. The kernel of the irreducible representation ρ_1 is $K_1 = \langle a \rangle$, of order 4. Since A_1 is Prym-Tyurin for C and $e(A_1) = 4 =$

$|K_1|$, by Theorem 43, f_1 is a normalization and thus,

$$deg(\varphi_1) = deg(\psi_1) = 4.$$

Now, A_4 is a 2-dimensional variety of type (5 5) given by

$$\frac{\langle 4\epsilon_1, 4\epsilon_2, 4\delta_1, 4\delta_2\rangle_{\mathbb{C}}}{\langle 4\epsilon_1, 4\epsilon_2, 4\delta_1, 4\delta_2\rangle_{\mathbb{Z}}},$$

with

$$\epsilon_1 = \frac{\alpha_1 - 2\alpha_3}{4} \text{ and } \epsilon_2 = \frac{\alpha_2 - 2\alpha_4}{4};$$

$$\beta_1 = \frac{\beta_1 - 2\beta_3}{4} \text{ and } \beta_2 = \frac{\beta_2 - 2\beta_4}{4}.$$

The irreducible representation ρ_4 has trivial kernel and consequently, $deg(\psi_4) = 1$ and $\tilde{C}_4 = C$. Moreover, as A_4 is Prym-Tyurin of exponent $e(A_4) = 5$ for C, by [4, Lemma 1.3] $deg(\varphi_4)$ divides $e(A_4)$. Thus $deg(\varphi_4) = 1$ or 5. Assume first that $deg(\varphi_4) = 5$. Then by Proposition 36, C_4 is smooth and $A_4 = J(C_4)$. Hence $g(C_4) = dim(A_4) = 2$, by Riemman Hurwitz applyed to φ_4, we have

$$2g(C) - 2 = deg(\varphi_4)(2g(C_4) - 2) + R$$

and thus $R = -4$, an absurd. Thus, $deg(\varphi_4) \neq 5$ and we must have

$$deg(\varphi_4) = 1.$$

5.4 Action on a family of curves

We now consider the group $G_m = \langle a, b |\ a^{2^m} = b^2 = baba^{-d} = 1\rangle$ such that $d = 2^{m-1} - 1$ and $|G_m| = 2^{m+1}$ and $m \geq 3$. Let C be a curve of genus $g(C) = 2^{m-2}$ given by the equation

$$y^2 = x(x^{2^{m-1}} - 1).$$

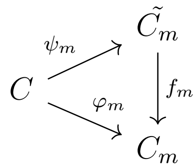
If G_m acts on C by

$$a(x, y) = (\xi^2 x, \xi y) \text{ and } b(x, y) = \left(\frac{1}{\xi^2 x}, \frac{-i\xi^d y}{x^{2(m-2)} + 1}\right)$$

where ξ is a primitive 2^m-th root of unity, we have the isotypical decomposition of the Jacobian variety given by

$$J(C) \simeq A_m^2,$$

with $A_m = J(\tilde{C_m})$, where $\tilde{C_m} = \dfrac{C}{< b >}$ and $g(\tilde{C_m}) = 2^{m-3}$. Since we have the morphisms $\psi_m : C \to \tilde{C_m}$ and $\varphi_m : C \to C_m$, there is a morphism $f_m : \tilde{C_m} \to C_m$ such that the following diagram



is commutative, where $\varphi_m = \varphi_{A_m}$ and $C_m = \varphi_m(C)$. Furthermore, as the kernel of the representation of G_m associated to the factor A_m of the decomposition is $K_m = \langle b \rangle$ and $deg(\psi_m) = |K_m|$, we have $deg(\psi_m) = 2$. Applying Riemann Hurwitz's Theorem to $\psi_m : C \to \tilde{C_m}$ we see that ψ_m is not étale, since $m \geq 3$. Hence, by Propositions 15 and 37, follows that A_m is a Prym-Tyurin variety for C of exponent $deg(\psi_m)$, that is, $e(A_m) = 2$. On the other hand

$$deg(\varphi_m) = deg(f_m)deg(\psi_m) = 2deg(f_m)$$

and $deg(\varphi_m) \leq 2$, by Proposition 35. This implies that

$$deg(\varphi_m) = 2.$$

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