Plane Algebroid Curves in Arbitrary Characteristic

Curvas Algebróides Planas em Característica Arbitrária

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To my mom, Tania Eliana Garcia Agudo

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Abstract

The subject of this Dissertation is the study of germs of plane curves defined over arbitrary algebraically closed fields. Classically, this was performed over the field of complex numbers, by using as a main tool the Newton-Puiseux parametrization, related to the normalization of the curve. The theory was then adapted to arbitrary algebraically closed field using the so-called Hamburger-Noether expansions that take track of the entire desingularization process of the curve. In this work, we will use, instead, the notion of contact order among irreducible curves by means of the logarithmic distance introduced by J. Chadzynski and A. Ploski in [CP]. This attack works in arbitrary characteristic and avoids the use of the Hamburger-Noether expansions, making proofs simpler and more elegant.

The content of this dissertation is as follows:

In Chapter 1, we introduce the notion of algebroid plane curves, their normalization and their intersection theory. We used as a reference for this part the book of A. Seidenberg [Sei] and the survey of A. Hefez [He]. In Chapter 2 and 3, we introduce the notion of semigroup of values of an irreducible plane curve and make a detailed study of their properties, introducing at the end the important notion of Key-polynomials, showing that they are nothing else but some special Apéry polynomials. This part is based on [He] and personal notes of this author. In Chapter 4, we introduce the contact order among irreducible plane curves and study its properties, applying them to deduce some results about irreducible plane curves that have high contact order. The whole theory is used to deduce Merle's and Granja's theorems [Me] and [Gr] over arbitrary algebraically closed fields. To conclude the work we present a result due to E. Garcia Barroso and A. Ploski about the relation among the Milnor number of an irreducible power series and the conductor of its semigroup of values. In this part, we used the works of E. Garcia Barroso and A. Ploski [GB-P1] and [GB-P2].

Keywords: Singularities in positive characteristic, Milnor number in positive characteristic, Singularities of algebroid curves

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Resumo

O assunto dessa dissertação é o estudo dos germes de curvas planas definidas sobre corpos algebicamente fechados arbitrários. Classicamente tal estudo era realizado sobre o corpo dos números complexos, utilizando-se como principal ferramenta para isso as parametrizações de Newton-Puiseux, relacionadas com a normalização da curva. Em seguida, a teoria foi adaptada para corpos algebricamente fechados arbitrários, utilizando-se as chamadas expansões de Hamburger-Noether que levam em conta todo o processo de resolução da singularidade da curva. Neste trabalho, usaremos ao invés a noção de contato entre curvas irredutíveis por meio da distância logarítmica introduzida por J. Chadzynski e A. Ploski em [CP]. Essa abordagem funciona em característica arbitrária e evita o uso das expansões de Hamburger-Noether, tornando as demonstrações mais simples e elegantes.

O conteúdo dessa dissertação é o seguinte:

No Capítulo 1, introduzimos a noção de curvas algebróides planas, suas normalizações e a sua teoria de interseção. Usamos nessa parte como referência o livro de A. Seidenberg [Sei] e o ”survey de A. Hefez [He]. Nos Capítulos 2 e 3, introduzimos a noção de semigrupo de valores de uma curva plana irredutível e empreendemos um estudo detalhado de suas propriedades, introduzindo no final a importante noção de polinômios-chave, mostrando que não são nada além de polinômios de Apéry particulares. Nessa parte, baseamos-nos em [He] e em notas pessoais desse autor. No Capítulo 4, introduzimos a ordem de contato entre curvas irredutíveis planas e estudamos as suas propriedades, utilizando-as para deduzir alguns resultados sobre curvas irredutíveis que possuem ordem de contato alta. Toda essa teoria é utilizada para deduzir os teoremas de Merle e de Granja, contidos em [Me] e [Gr], sobre corpos algebricamente fechados arbitrários. Para concluir o trabalho, apresentamos um resultado recente devido a E. Garcia Barroso e A. Ploski sobre a relação entre o número de Milnor de uma série irredutvel e o condutor de seu semigrupo de valores. Nessa parte, utilizamos os trabalhos de E. Garcia Barroso e A. Ploski [GB-P1] e [GB-P2].

Contents

Chapter 1

Algebroid Plane Curves

In this chapter we will introduce the objects that will be studied in this dissertation, namely, the algebroid plane curves defined over arbitrary algebraically closed fields and their intersection theory.

1.1 Power series

The theory of algebroid plane curves relies on the notions of power series in one or two variables, as defined below.

Let K be an algebraically closed field of arbitrary characteristic $p \geq 0$ and let t, X and Y be indeterminates over K. We will denote by $K[[t]]$ (respectively, by $K[[X, Y]]$) the ring of formal power series in one indeterminate t (respectively, in two indeterminates X and Y), with coefficients in K. We will briefly recall some of their properties and refer to [He] for the proofs.

The ring $K[[t]]$ has elements of the form $\sum_{i=0}^{\infty} a_i t^i$, while the elements of $K[[X, Y]]$ are all formal sums of the type $f = \sum_{i=0}^{\infty} P_i$, where each P_i is a homogeneous polynomial of degree i in the indeterminates X and Y with coefficients in K. The zero polynomial will be considered to be a homogeneous polynomial of any degree. The ring operations are the usual addition and multiplication of power series.

If $f = \sum_{i=0}^{\infty} a_i t^i \in K[[t]]$, then mult($f$) is the least i such that $a_i \neq 0$. On the other hand, if $f = \sum_{i=0}^{\infty} P_i \in K[[X, Y]] \setminus \{0\}$, then $n = \text{mult}(f)$ is the least i such that $P_i \neq 0$. The homogeneous polynomial P_n is called the *initial form* of f and the integer n is called the *multiplicity* of f. If $f = 0$, we put $\text{mult}(f) = \infty$. The notion of multiplicity for power series plays a role similar to that of the degree for polynomials.

The multiplicity of power series has the following properties:

1. $\text{mult}(fg) = \text{mult}(f) + \text{mult}(g)$;

2. $\mathrm{mult}(f+g) \geq \min\{\mathrm{mult}(f),\ \mathrm{mult}(g)\}$, with equality sign holding whenever $\mathrm{mult}(f) \neq \mathrm{mult}(g)$.

3. $\mathrm{mult}(f) = 0$ if and only if f is a unit.

The rings $K[[t]]$ and $K[[X,Y]]$ are unitary commutative rings with a unique maximal ideal $\langle t\rangle$ and $\mathcal{M} = \langle X, Y\rangle$, respectively. The ring $K[[t]]$ is a *principal ideal domain*.

The automorphisms of $K[[t]]$ are of the form $t \mapsto a_1 t + a_2 t^2 + \cdots$, with $a_1 \neq 0$, while those of $K[[X,Y]]$ are of the form

$$\begin{aligned} X &\mapsto aX + bY + \cdots \\ Y &\mapsto cX + dY + \cdots , \end{aligned}$$

where $ad - bc \neq 0$.

Two elements f and g in a ring R are said *associated* if there exists a unit $u \in R$ such that $f = ug$.

We will say that $f \in K[[X,Y]]$ is regular in the indeterminate Y of order n, if $f(0,Y) = Y^n u(Y)$, where $u(Y)$ is a unit in $K[[Y]]$.

Let us recall the following result:

Theorem 1.1 (The Division Theorem). *Let $f \in \mathcal{M}$ be regular in Y of order n. Given any $g \in K[[X,Y]]$ there exist $q \in K[[X,Y]]$ and $r \in K[[X]][Y]$ with $r = 0$ or $\deg_Y r < n$, uniquely determined by f and g, such that*

$$g = fq + r$$

This result implies the following one:

Theorem 1.2 (The Weierstrass Preparation Theorem). *Let $f \in K[[X,Y]] \setminus K$ be of multiplicity n. Then there exist a K-automorphism ϕ of $K[[X,Y]]$, a unit $u \in K[[X,Y]]$ and $a_1, \ldots, a_n \in K[[X]]$, with $\mathrm{mult}(a_i) \geq i$ for $i = 1, \ldots, n$, such that*

$$\phi(f)u = Y^n + a_1 Y^{n-1} + \cdots + a_n.$$

A polynomial of the form

$$Y^n + a_1 Y^{n-1} + \cdots + a_n \in K[[X]][Y],$$

where $a_i(0) = 0$, for $i = 1, \ldots, n$, will be called a *distinguished polynomial* or simply a *d-polynomial*. If in addition, we have $\mathrm{mult}(a_i) \geq i$, for all i, it will be called a *Weierstrass polynomial* or simply a *w-polynomial*. Notice that a w-polynomial is regular in Y of order equal to its multiplicity.

If $f \in K[[X]][Y]$ is a d-polynomial, then f is reducible in $K[[X,Y]]$ if and only if f is reducible in $K[[X]][Y]$. This, together with the Weierstrass Preparation Theorem, imply that $K[[X,Y]]$ is a unique factorization domain.

1.2 Algebroid plane curves

An *algebroid plane curve* is the equivalence class (f) of associated power series to a given power series $f \in \mathcal{M} = \langle X, Y \rangle \subset K[[X, Y]]$.

Since the multiplicity of a formal power series remains invariant when we multiply it by unit, we may define the multiplicity of an algebroid plane curve (f) as being the multiplicity of f. An algebroid curve of multiplicity one will be called *smooth*. When the multiplicity is greater than one, we will say that the curve is *singular*.

Let (f) be an algebroid plane curve. We say that the curve (f) is *irreducible* if the formal power series f is irreducible in $K[[X, Y]]$. Notice that this notion is independent of the representative f of (f). An irreducible algebroid plane curve will also be called a *plane branch* or shortly a *branch*.

Let (f) be an algebroid plane curve and consider the decomposition of f into irreducible factors in $K[[X, Y]]$

$$f = f_1 \cdots f_r.$$

The algebroid plane curves (f_j), for $j = 1, \ldots, r$, above defined, are called the branches of the curve (f). The curve (f) will be called reduced if $(f_i) \neq (f_j)$ for $i \neq j$, that is, when f_i and f_j are not associated if $i \neq j$.

Most properties of an algebroid plane curve are preserved after we change coordinates in $K[[X, Y]]$ through a K-automorphism. This motivates the next fundamental definition.

Two algebroid plane curves (f) and (g) will be said equivalent, writing in such case $(f) \sim (g)$, if there exists a K-automorphism ϕ of $K[[X, Y]]$ such that

$$(\phi(f)) = (g).$$

Since any branch is equivalent to a curve defined by a w-polynomial, when convenient, we may suppose that its equation is a w-polynomial.

Given an algebroid plane curve (f) of multiplicity n, that is, $f = P_n + P_{n+1} + \cdots$, where each P_i is a homogeneous polynomial in $K[X, Y] \subset K[[X, Y]]$ of degree i and $P_n \neq 0$, then the curve (P_n) is uniquely determined by the curve (f) and will be called the *tangent cone* of the curve (f).

Since any homogeneous polynomial in two indeterminates with coefficients in an algebraically closed field decomposes into linear factors, we may write

$$P_n = \prod_{i=1}^{s} (a_i X + b_i Y)^{r_i}$$

where $\sum_{i=1}^{s} r_i = n$, $a_i,\ b_i \in K$, for $i,\ j = 1, \ldots, s$, and $a_ib_j - a_jb_i \neq 0$ if $i \neq j$. So, the tangent cone of (f) consists of the lines $(a_iX + b_iY)$, $i = 1, \ldots, s$, each counted with multiplicity r_i, called the *tangent lines* of (f).

It is known that a branch (f) has a unique tangent line, counted with multiplicity equal to $\mathrm{mult}(f)$.

Let f be an element in the maximal ideal $\mathcal{M} = \langle X, Y\rangle$ of $K[[X, Y]]$, and let $\langle f\rangle$ be the ideal generated by f. We define the *coordinates ring* of the curve (f) as being the K-algebra

$$\mathcal{O}_f = \frac{K[[X, Y]]}{\langle f\rangle}.$$

We will denote the residual class of Y by y and the residual class of X by x.

The ring $\mathcal{O}_f$ is a local ring with maximal ideal $\mathcal{M}_f = \overline{\mathcal{M}}$. When f is irreducible, the ideal $\langle f\rangle$ is prime and $\mathcal{O}_f$ is an integral domain. In this case, the field of fractions of $\mathcal{O}_f$ will be denoted by $\mathcal{K}_f$. The next result will tell us that the ring $\mathcal{O}_f$ is an important invariant of the equivalence classes of algebroid plane curves.

Let (f) and (g) be two algebroid plane curves. We have that $(f) \sim (g)$ if and only if $\mathcal{O}_f$ and $\mathcal{O}_g$ are isomorphic as K-algebras.

Another important structure of $\mathcal{O}_f$ is the following:

Suppose that $f \in K[[X, Y]]$ is regular in Y of order n, then $\mathcal{O}_f$ is a free $K[[X]]$-module of rank n generated by the residual classes y^i of the Y^i, $i = 0, \ldots, n-1$, in $\mathcal{O}_f$. In other words,

$$\mathcal{O}_f = K[[X]] \oplus K[[X]]y \oplus \cdots \oplus K[[X]]y^{n-1}.$$

1.3 Intersection of curves

Let $f,\ g \in \mathcal{M}$. The following conditions are equivalent:

i) f and g are relatively prime;

ii) The dimension of $\frac{K[[X,Y]]}{\langle f,g\rangle}$ as a K-vector space is finite.

The *intersection index* of f and g is the integer (including ∞)

$$\mathrm{I}(f, g) = \dim_K \frac{K[[X, Y]]}{\langle f, g\rangle}$$

Notice that if f or g is a unit in $K[[X, Y]]$, then $\langle f, g\rangle = K[[X, Y]]$ and therefore $\mathrm{I}(f, g) = 0$.

We will say that two algebroid curves (f) and (g) are *transversal* if (f) and (g) are smooth and their tangent lines are distinct.

Let $f,\ g,\ h,\ u,\ v \in K[[X,Y]]$, with u and v units. and ϕ an automorphism of $K[[X,Y]]$. The intersection index has the following properties:

i) $\mathrm{I}(f,g) < \infty$ if and only if f and g are relatively prime in $K[[X,Y]]$;

ii) $\mathrm{I}(f,g) = \mathrm{I}(g,f)$;

iii) $\mathrm{I}(\phi(f),\phi(g)) = \mathrm{I}(uf,vg) = \mathrm{I}(f,g)$;

iv) $\mathrm{I}(f,hg) = \mathrm{I}(f,g) + \mathrm{I}(f,h)$;

v) $\mathrm{I}(f,g) = 1$ if and only if (f) and (g) are smooth with distinct tangents;

vi) $\mathrm{I}(f,g-hf) = \mathrm{I}(f,g)$.

Let (f) be an irreducible algebroid curve. A *parametrization* of (f) is a pair $(\phi(t),\psi(t)) \in K[[t]]^2$ such that $\phi(t) \neq 0$ or $\psi(t) \neq 0$, $\phi(0) = \psi(0) = 0$ and $f(\phi(t),\psi(t)) = 0$. We say that the parametrization $(\phi(t),\psi(t))$ is a *good parametrization* if the field of fractions of the ring $K[[\phi(t),\psi(t)]]$ is equal to the field of fraction $K((t))$ of the ring $K[[t]]$.

The following results are fundamental.

Theorem 1.3 (Normalization Theorem). *If $f = f(X,Y) \in K[[X,Y]]$ is an irreducible power series, then there exists a good parametrization $(\phi(t),\psi(t))$ of (f). Moreover, if $(\alpha(s),\beta(s)) \in K[[s]]$ is a parametrization of (f), then there exists a power series $\sigma(s) \in K[[s]]$ such that $\sigma(0) = 0$, $\alpha(s) = \phi(\sigma(s))$ and $\beta(s) = \psi(\sigma(s))$.*

From this it follows that if $(\phi(t),\psi(t))$ and $(\alpha(s),\beta(s))$ are both good parametrizations of (f), then the map $t \mapsto \sigma(s)$ is an isomorphism from $K[[t]]$ onto $K[[s]]$.

Theorem 1.4. *Let $f = Y^d + a_1(X)Y^{d-1} + \cdots + a_0(X) \in K[[X]][Y]$. Then for some $\phi(t) \in K[[t]]$, $f(\phi(t),Y)$ splits completely into linear factors in $K[[t]][Y]$.*

We describe below another way to compute the intersection index among curves.

Let $f(X,Y) \in K[[X,Y]]$ be an irreducible power series and $(\phi(t),\psi(t))$ a good parametrization of (f). Then for any power series $g = g(X,Y)$ in $K[[X,Y]]$ we have

$$\mathrm{I}(f,g) = \mathrm{mult}(g(\phi(t),\psi(t))).$$

Taking $g = X$ (respectively, $g = Y$) we get from the above formula that $\mathrm{mult}(f(0,Y)) = \mathrm{I}(f,X) = \mathrm{mult}(\phi(t))$ and $\mathrm{mult}(f(X,0)) = \mathrm{I}(f,Y) = \mathrm{mult}(\psi(t))$.

If f is irreducible and $\mathrm{I}(f,g) = \mathrm{I}(f,h) < \infty$, then there exists a constant $c \in K$ such that

$$\mathrm{I}(f, g - ch) > \mathrm{I}(f,g).$$

Theorem 1.5. *Let $f(X,Y) \in K[[X,Y]]$ be an irreducible power series such that $f(0,Y) \neq 0$ and let $(\alpha(t),\beta(t))$ with $\alpha(t) \neq 0$ be a parametrization of (f), then for every power series $g(X,Y) \in K[[X,Y]]$ we have*

$$\mathrm{mult}(g(\alpha(t),\beta(t))) = \frac{\mathrm{I}(f,g)}{\mathrm{I}(f,X)}\,\mathrm{mult}(\alpha(t)).$$

Given a good parametrization $(\phi(t),\psi(t))$ of an irreducible algebroid curve (f), we define a numerical function as follows:

$$\begin{array}{rcl} \nu_f\colon \mathcal{O}_f \setminus \{0\} & \to & \mathbb{N} \cup \{\infty\}, \\ g & \mapsto & \mathrm{mult}(g(\phi(t),\psi(t))) \end{array}$$

where $\mathbb{N}$ denotes the set of non negative integers. One has $\nu_f(g) = \infty$ if and only if $g = 0$.

This numerical function is independent from the choice of the good parametrization since, in this case, as we mentioned above, $\mathrm{mult}(g(\phi(t),\psi(t))) = \mathrm{I}(f,g)$.

From the fact that $\nu_f(g) = \mathrm{I}(f,g)$, the function ν_f has the following properties:

For all $g_1, g_2 \in \mathcal{O}_f$, one has

1. $\nu_f(g_1 g_2) = \nu_f(g_1) + \nu_f(g_2)$;
2. $\nu_f(u) = 0$ if, and only if, u is a unit in $\mathcal{O}_f$;
3. $\nu_f(g_1 + g_2) \geq \min\{\nu_f(g_1), \nu_f(g_2)\}$ with equality verified whenever $\nu_f(g_1) \neq \nu_f(g_2)$.

Theorem 1.6. *Let $f,\ g \in \mathcal{M}$. We have that*

$$\mathrm{I}(f,g) \geq \mathrm{mult}(f)\,\mathrm{mult}(g)$$

with the equality verified if, and only if, (f) and (g) have no common tangents.

Chapter 2

Arithmetical Semigroups

Semigroups play an important role in the theory of algebroid curves, as we will see in the course of this dissertation. Much of the presentation here in this chapter was influenced by the article of G. Angermuller [An].

2.1 Semigroups

Let $\{0\} \subsetneq G \subset \mathbb{N}$. We say that G is a numerical *semigroup* if it is closed under addition.

Our main concern will be with semigroup associated to an irreducible curve (f) as defined below:

$$G_f = \{\mathrm{I}(f, h);\ h \in K[[X, Y]] \setminus \langle f \rangle\}.$$

That G_f is a semigroup follows immediately from the additivity of the intersection index.

In this chapter we will initiate the study of semigroups under an arithmetical point of view.

The element $\min(G \setminus \{0\})$ is called the *multiplicity* of G, and will be denoted by $\mathrm{mult}(G)$.

If $v_0, \ldots, v_g \in \mathbb{N}$, then

$$G = \langle v_0, \ldots, v_g \rangle = \{\lambda_0 v_0 + \cdots + \lambda_g v_g;\ \lambda_0, \ldots, \lambda_g \in \mathbb{N}\}$$

is clearly a numerical semigroup, called the semigroup generated by $v_0, \ldots, v_g$, which in turn are called the *generators* of G.

Proposition 2.1. *Given any numerical semigroup G, there exist a finite set of elements $v_0, \ldots, v_g$ in G such that*

i) $v_0 < \cdots < v_g$, *and* $v_i \not\equiv v_j \bmod v_0$, *for* $i \neq j$*;*
ii) $G = \langle v_0, \ldots, v_g \rangle$*;*
iii) $\{v_0, \ldots, v_g\}$ *is contained in any set of generators of* G*.*

Proof. We are going to define $v_0, \ldots, v_g$ as follows:

$$v_0 = \mathrm{mult}(G) \text{ and } v_i = \min(G \setminus \langle v_0, \ldots, v_{i-1} \rangle), \; i = 1, \ldots, g.$$

i) Suppose that $i \neq j$. We may assume that $i < j$. Then it is clear that $v_i \not\equiv v_j \bmod v_0$ for $i < j$ because, otherwise, v_j would be in $\langle v_0, \ldots, v_i \rangle$, which is a contradiction. Notice that $v_i, \; v_{i+1} \in G \setminus \langle v_0, \ldots, v_{i-1} \rangle$ so, $v_i < v_{i+1}$, because v_i is the least element with this property. This shows that $v_0 < \cdots < v_g$.
ii) Since $v_i \not\equiv v_j \bmod v_0$ for $i \neq j$, then for some $g < v_0$ this process must stop. Therefore $G = \langle v_0, \ldots, v_g \rangle$.
iii) Let $\{u_0, \ldots, u_r\}$ be a set of generators of G, so

$$v_i = \lambda_{i,0} u_0 + \cdots + \lambda_{i,r} u_r, \text{ with } \lambda_{i,0}, \ldots, \lambda_{i,r} \in \mathbb{N}, \; i \in \{0, \ldots, g\}.$$

On the other hand,

$$u_j = \alpha_{0,j} v_0 + \cdots + \alpha_{g,j} v_g, \text{ with } \alpha_{0,j}, \ldots, \alpha_{g,j} \in \mathbb{N}, \; j \in \{0, \ldots, r\}.$$

Then,

$$\begin{aligned} v_i &= \lambda_{i,0} u_0 + \cdots + \lambda_{i,r} u_r \\ &= \lambda_{i,0}(\alpha_{0,0} v_0 + \cdots + \alpha_{g,0} v_g) + \cdots + \lambda_{i,r}(\alpha_{0,r} v_0 + \cdots + \alpha_{g,r} v_g) \\ &= (\lambda_{i,0}\alpha_{0,0} + \cdots + \lambda_{i,r}\alpha_{0,r}) v_0 + \cdots + (\lambda_{i,0}\alpha_{i,0} + \cdots + \lambda_{i,r}\alpha_{i,r}) v_i + \\ &\quad + \cdots + (\lambda_{i,0}\alpha_{g,0} + \cdots + \lambda_{i,r}\alpha_{g,r}) v_g. \end{aligned}$$

Since $v_i \notin \langle v_0, \ldots, v_{i-1} \rangle$ then some of the coefficients of $v_i, v_{i+1}, \ldots, v_g$ must be nonzero. Since $v_i < v_{i+1} < \cdots < v_g$, it must be that of v_i and the others are zero. This implies that for some $l \in \{0, \ldots, r\}$, $\lambda_{i,l} = 1$ and the others $\lambda_{i,j}$ are zero. This shows that $v_i = u_l$. □

The set $\{v_0, \ldots, v_g\}$ of Proposition 2.1 will be called the *minimal system of generators* of G and the integer g will be called the *genus* of the semigroup. Notice that from Proposition 2.1 (i) one has that $g \leq \mathrm{mult}(G) - 1$.

2.2 Semigroups with conductor

Given a numerical semigroup G, then the elements of $\mathbb{N} \setminus G$ are called the *gaps* of G. A semigroup may have finitely or infinitely many gaps.

When the number of gaps is finite then there exists an element $c \in G$, called the *conductor* of G, such that

a) $c - 1 \notin G$;

b) if $z \in \mathbb{N}$ and $z \geq c$ then $z \in G$.

Proposition 2.2. *Let G be a numerical semigroup. The following assertions are equivalent:*

i) *G has a conductor;*

ii) *The elements of G have* gcd *equal to one;*

iii) *There exist two consecutive integers in G.*

Proof. i) $\Rightarrow$ ii) If G has a conductor then $\gcd(G) = 1$, since $G \subset \langle \gcd(G) \rangle$.
ii) $\Rightarrow$ iii) Let $v_0, \ldots, v_g$ be the minimal system of generators of G. Then $\gcd(G) = 1$ implies $\gcd(v_0, \ldots, v_g) = 1$. So there exist integers $\lambda_0, \ldots, \lambda_g$ such that

$$\lambda_0 v_0 + \cdots + \lambda_g v_g = 1,$$

transferring to the right hand side of the equality the negative terms, the result follows immediately.
iii) $\Rightarrow$ i) Let a and $a + 1$ be two elements in G then the set

$$\{0,\ a + 1,\ 2(a + 1), \ldots, (a - 1)(a + 1)\}$$

is a complete residue system modulo a. So, any integer $n \geq (a - 1)(a + 1)$ may be written as $n = \lambda(a + 1) + \mu a$ with $0 \leq \lambda < a$. So, $\mu \geq 0$ and consequently $n \in G$. □

Remark 2.3. *Notice that in the proof of the Proposition 2.2 we got the estimate $c \leq (a - 1)(a + 1)$ for the conductor of G, when a and $a + 1$ are elements of G.*

Example 2.4. *Important examples of numerical semigroups with conductors are the semigroups G_f associated to a plane branch (f).*

Indeed, let $(\phi(t), \psi(t))$ be a good parametrization of the branch (f), then we have $K((t)) = K((\phi(t), \psi(t)))$, so there exist $P = P(X, Y), Q = Q(X, Y) \in K[[X, Y]]$, with $Q \not\equiv 0 \bmod f$, such that

$$t = \frac{P(\phi(t), \psi(t))}{Q(\phi(t), \psi(t))}.$$

It then follows that

$$\mathrm{I}(f, P) = \mathrm{mult}(P(\phi(t), \psi(t))) = \mathrm{mult}(Q(\phi(t), \psi(t))) + 1 = \mathrm{I}(f, Q) + 1,$$

implying that in G_f there are two consecutive integers, which in view of Proposition 2.2, allows us to conclude that G_f has a conductor.

Let $G = \langle v_0, \dots, v_g \rangle$ be a numerical semigroup with conductor. We define below two sequences of numbers associated to the set of generators of G. Put $e_0 = v_0$, $n_0 = 1$, and for $i = 1, \dots, g$,

$$e_i = \gcd(v_0, \dots, v_i) \text{ and } n_i = \frac{e_{i-1}}{e_i}.$$

Remark 2.5. *From the definition of the e_i's it is clear that $e_i | e_{i-1}$, so $n_i \in \mathbb{N}$ for all $i = 1, \dots, g$. Also, $e_g = \gcd(v_0, \dots, v_g) = 1$. We also have $n_0 \cdots n_i e_i = v_0$. In particular, $n_0 \cdots n_g = v_0$ and $n_{i+1} \cdots n_g = e_i$.*

Given a numerical semigroup $G = \langle v_0, \dots, v_g \rangle$, then any element in G may be represented in several ways in the form

$$\lambda_0 v_0 + \cdots + \lambda_g v_g, \quad \lambda_0, \dots, \lambda_g \in \mathbb{N}.$$

But, when G is a semigroup with conductor we will show in Proposition 2.7 below that the elements of G may be represented uniquely as a combination of special type of the elements $v_0, \dots, v_g$.

Lemma 2.6. *Let $v_0, \dots, v_g \in \mathbb{N}$ with $\gcd(v_0, \dots, v_g) = 1$ and let e_i and n_i, $i = 0, \dots, g$, be their associated integers. For every $m \in \mathbb{N}$ there is a unique solution for the congruence*

$$m \equiv \sum_{i=1}^{g} s_i v_i \bmod v_0, \text{ with } 0 \leq s_i < n_i, \ i = 1, \dots, g.$$

Proof. By induction on g. If $g = 1$ we have $e_1 = \gcd(v_0, v_1) = 1$ and $n_1 = v_0$, then there exist integers λ and μ such that $\lambda v_0 + \mu v_1 = 1$. So,

$$m\lambda v_0 + m\mu v_1 = m.$$

Dividing $m\mu$ by v_0 we get

$$m\mu = q v_0 + s_1 \text{ with } q \in \mathbb{Z} \text{ and } 0 \leq s_1 < v_0.$$

Thus,

$$m = m\lambda v_0 + m\mu v_1 = m\lambda v_0 + (q v_0 + s_1) v_1 = s_1 v_1 + (m\lambda + q) v_0 \equiv s_1 v_1 \bmod v_0.$$

Let us now suppose the result true for $g \geq 1$ and let $v_0, \dots, v_{g+1}$ be positive integers satisfying the hypotheses of the lemma. Consider the sequence $v_0' = \frac{v_0}{e_g}, \dots, v_g' = \frac{v_g}{e_g}$ and notice that $\gcd(v_0', \dots, v_g') = 1$. So, by the inductive hypothesis, for every integer m', there exist integers s_i and λ, such that

$$m' = \sum_{i=1}^{g} s_i v_i' + \lambda v_0', \text{ with } 0 \leq s_i < n_i', \ i = 1, \dots, g,$$

where
$$n'_i = \frac{e'_{i-1}}{e'_i} = \frac{\gcd(v'_0, \dots, v'_{i-1})}{\gcd(v'_0, \dots, v'_i)} = \frac{\gcd(v_0, \dots, v_{i-1})/e_g}{\gcd(v'_0, \dots, v'_i)/e_g} = n_i.$$

Since $v_0, \dots, v_{g+1}$ are relatively prime, there exist integers $\lambda_0, \dots, \lambda_{g+1}$ such that $1 = \lambda_0 v_0 + \cdots + \lambda_{g+1} v_{g+1}$, then for every integer m, we have

$$m = m\lambda_0 v_0 + \cdots + m\lambda_{g+1} v_{g+1}.$$

Dividing $m\lambda_{g+1}$ by $n_{g+1} = e_g$, we get

$$m\lambda_{g+1} = qn_{g+1} + s_{g+1} \text{ with } 0 \leq s_{g+1} < n_{g+1} \text{ and } q \in \mathbb{Z}.$$

Since e_g divides $v_0, \dots, v_g$ and $n_{g+1} = e_g$, there exists an integer m' such that

$$m'e_g = m\lambda_0 v_0 + \cdots + m\lambda_g v_g + qn_{g+1}.$$

Then

$$\begin{aligned} m &= m\lambda_0 v_0 + \cdots + m\lambda_g v_g + m\lambda_{g+1} v_{g+1} \\ &= m\lambda_0 v_0 + \cdots + m\lambda_g v_g + (qn_{g+1} + s_{g+1}) v_{g+1} \\ &= m'e_g + s_{g+1} v_{g+1} \\ &= \left(\sum_{i=1}^{g} s_i \frac{v_i}{e_g} + \lambda \frac{v_0}{e_g} \right) e_g + s_{g+1} v_{g+1}. \end{aligned}$$

So,

$$m \equiv \sum_{i=1}^{g+1} s_i v_i \bmod v_0, \quad \text{with } 0 \leq s_i < n_i, \ i = 1, \dots, g+1.$$

The uniqueness follows from the facts that every integer is congruent modulo v_0 to $\sum_{i=1}^{g} s_i v_i$, for some $0 \leq s_i < n_i$, $i = 1, \dots, g$, and that

$$\sharp \left\{ \sum_{i=1}^{g} s_i v_i \bmod v_0; \ 0 \leq s_i < n_i, \ i = 1, \dots, g \right\} = n_1 n_2 \cdots n_g = v_0.$$

□

Proposition 2.7. *Let $v_0, \dots, v_g$ be relatively prime natural numbers and let c be the conductor of the semigroup $G = \langle v_0, \dots, v_g \rangle$. Then*

i) *Every natural number m has a unique representation as*

$$m = \sum_{i=0}^{g} s_i v_i, \quad \textit{with } 0 \leq s_i < n_i, \ i = 1, \dots, g \textit{ and } s_0 \in \mathbb{Z}.$$

ii) $c \le \sum_{i=1}^{g}(n_i - 1)v_i - v_0 + 1$.

Proof. i) By Lemma 2.6 we have $m \equiv \sum_{i=1}^{g} s_i v_i \bmod v_0$ with $0 \le s_i < n_i,\ i = 1, \dots, g$; so

$$m = \sum_{i=0}^{g} s_i v_i, \quad \text{with } 0 \le s_i < n_i,\ i = 1, \dots, g \text{ and } s_0 \in \mathbb{Z}$$

and this representation is unique.
ii) Let m be an integer such that $m > \sum_{i=1}^{g}(n_i - 1)v_i - v_0 + 1$. From (i) we have a unique representation $m = \sum_{i=0}^{g} s_i v_i$ with $0 \le s_i < n_i,\ i = 1, \dots, g$ and $s_0 \in \mathbb{Z}$, thus

$$\sum_{i=0}^{g} s_i v_i = m > \sum_{i=1}^{g}(n_i - 1)v_i - v_0 + 1 \ge \sum_{i=1}^{g} s_i v_i - v_0,$$

which implies that $s_0 v_0 > -v_0$, so $s_0 \ge 0$. Therefore, $m \in G$. □

From Proposition 2.7, if $s_0 \ge 0$ in the representation of m, then $m \in G$. The converse is not always true, which means that we may have $m \in G$ and $s_0 < 0$, as we can see in the following example.

Example 2.8. *Let* $G = \langle 8, 10, 11 \rangle$. *We have* $e_0 = 8$, $e_1 = 2$ *and* $e_2 = 1$. *So,* $n_1 = 4$, $n_2 = 2$ *and* $1 - v_0 + (n_1 - 1)v_1 + (n_2 - 1)v_2 = 1 - 8 + (4 - 1)10 + (2 - 1)11 = 34 > 26 = c$ *(the conductor of G). The element* $22 = 2 \cdot 11$ *is in G and, in the above representation, it is written as* $22 = 3v_1 + 0v_2 - v_0$.

Now, we are going to define the notion of nice sequence. Let $v_0, \dots, v_g$ be relatively prime non-negative integers and let e_i and n_i, $i = 1, \dots, g$, their associated integers. We say that the sequence $v_0, \dots, v_g$ is *nice* if, for all $i = 1, \dots, g$, we have

$$n_i v_i \in \langle v_0, \dots, v_{i-1} \rangle.$$

In a numerical semigroup G generated by a nice sequence, we have, with the above notation, that if $m \in G$ then $s_0 \ge 0$ and we get a formula for the conductor of G. This fact will be shown in the following proposition.

Proposition 2.9. *Let* $v_0, \dots, v_g$ *be a nice sequence of integers. If* $G = \langle v_0, \dots, v_g \rangle$ *and* e_i'*s and* n_i'*s are their associated integers, then*

i) *An integer* $m = \sum_{i=0}^{g} s_i v_i$, *with* $0 \le s_i < n_i,\ i = 1, \dots, g$, *and* $s_0 \in \mathbb{Z}$ *belongs to G if and only if* $s_0 \ge 0$.

ii) *The conductor c of G is given by*

$$c = \sum_{i=1}^{g}(n_i - 1)v_i - v_0 + 1.$$

Proof. i) Let $m = \lambda_0 v_0 + \cdots + \lambda_g v_g \in G$ with $\lambda_i \in \mathbb{N}$, $i = 0, \ldots, g$. Dividing λ_g by n_g we have $\lambda_g = q_g n_g + s'_g$ with $0 \leq s'_g < n_g$. Since the sequence is nice, we get

$$m_1 = \lambda_0 v_0 + \cdots + \lambda_{g-1} v_{g-1} + q_g n_g v_g \in \langle v_0, \ldots, v_{g-1} \rangle.$$

This implies that

$$m_1 = \lambda'_0 v_0 + \cdots + \lambda'_{g-1} v_{g-1} \text{ for some } \lambda'_0, \ldots, \lambda'_{g-1} \in \mathbb{N}.$$

Now we repeat this procedure with λ'_{g-1} and so on. This shows that one may write $m = \sum_{i=0}^{g} s'_i v_i$, with $0 \leq s'_i < n_i$, $i = 1, \ldots, g$, and $s'_0 \in \mathbb{N}$, so, by the uniqueness in Proposition 2.7 one has that $s_i = s'_i$, for all i, hence $s_0 = s'_0 \in \mathbb{N}$

ii) From Proposition 2.7 we know that $c \leq \sum_{i=1}^{g} (n_i - 1) v_i - v_0 + 1$. On the other hand, from (i), we have that $\sum_{i=1}^{g} (n_i - 1) v_i - v_0 \notin G$, so the equality holds. □

Another remarkable property that some numerical semigroups have is symmetry, in the following sense:

A numerical semigroup G with conductor c will be called *symmetric* if

$$\text{For all } z \in \mathbb{N},\ z \in G \Leftrightarrow\ c - 1 - z \notin G.$$

Notice that the implication $z \in G \Rightarrow c - 1 - z \notin G$ is true in any numerical semigroup with conductor. In fact if z and $c - 1 - z$ are in G, then $c - 1 \in G$, which is a contradiction.

Proposition 2.10. *Let G be a numerical semigroup with conductor c. The following assertions are equivalent:*

i) *G is symmetric;*

ii) *$2\,\sharp\,(G \cap [0, c)) = c$;*

iii) *$2\,\sharp\,(\mathbb{N} \setminus G) = c$;*

iv) *$\sharp\,(G \cap [0, c)) = \sharp\,(\mathbb{N} \setminus G)$.*

Proof. Notice that $(G \cap [0, c)) \cap (\mathbb{N} \setminus G) = \emptyset$ and $(G \cap [0, c)) \cup (\mathbb{N} \setminus G) = [0, c)$. This implies that (ii), (iii) and (iv) are equivalent.

Consider now the bijection

$$\begin{array}{rccc} \phi\colon & [0, c-1] & \to & [0, c-1] \\ & z & \mapsto & c - 1 - z. \end{array}$$

We have that G is symmetric if and only if $z \in G \Leftrightarrow c - 1 - z \notin G$ if and only if $\phi(G \cap [0, c)) = \mathbb{N} \setminus G$.

It follows that if (i) is satisfied, then $\sharp\,(G \cap [0, c)) = \sharp\,\phi(G \cap [0, c)) = \sharp(\mathbb{N} \setminus G)$, and consequently (iv) is satisfied.

Conversely, if (iv) is satisfied, and since $\phi(G \cap [0, c)) \subseteq \mathbb{N} \setminus G$, it follows that $\phi(G \cap [0, c)) = \mathbb{N} \setminus G$, then G is symmetric. □

Remark 2.11. i) *If G is symmetric, then c is even. Moreover, the symmetry of G is equivalent to the condition that there are as many gaps as non gaps in G.*
ii) *Let G and H be symmetric semigroups with the same conductor c and such that $H \subset G$ then $H = G$, indeed we have*

$$z \in G \Rightarrow c - 1 - z \notin G \Rightarrow c - 1 - z \notin H \Rightarrow z \in H.$$

Proposition 2.12. *Every semigroup generated by a nice sequence is symmetric.*

Proof. Let $v_0, \dots, v_g$ be a nice sequence. We know from Proposition 2.7 that the conductor of $G = \langle v_0, \dots, v_g \rangle$ is $c = \sum_{i=1}^{g} (n_i - 1)v_i - v_0 + 1$. We also know from Proposition 2.7 that any $z \in \mathbb{N}$ may be written uniquely as

$$z = \sum_{i=0}^{g} s_i v_i, \quad 0 \leq s_i < n_i, \ i = 1, \dots, g, \ s_0 \in \mathbb{Z}.$$

We also know, from Proposition 2.9, that $z \in G$ if and only if $s_0 \geq 0$. Then

$$c - 1 - z = \sum_{i=1}^{g} (n_i - 1)v_i - v_0 + 1 - 1 - \sum_{i=0}^{g} s_i v_i = \sum_{i=1}^{g} (n_i - 1 - s_i)v_i - (1 + s_0)v_0.$$

Hence we have

$$z \in G \Leftrightarrow s_0 \geq 0 \Leftrightarrow -(s_0 + 1) \leq -1 \ \Leftrightarrow c - 1 - z \notin G.$$

□

2.3 The Apéry sequence of a semigroup

Let G be a semigroup with conductor c and let m be any element in $G \setminus \{0\}$. We define the *Apéry sequence* of G with respect to m, inductively, as follows: $a_0 = 0$ and

$$a_j = \min \left(G \setminus \bigcup_{i=1}^{j-1} (a_i + m\mathbb{N}) \right), \ \ 1 \leq j \leq m - 1.$$

The following proposition holds.

Proposition 2.13. *The Apéry sequence of a semigroup G, with respect to $m \in G \setminus \{0\}$ satisfies the following properties, where $[a]$ is the residual class of a modulo m in $\mathbb{N}$.*

i) $0 = a_0 < a_1 < \cdots < a_{m-1}$;

ii) $a_i \not\equiv a_j \bmod m$ *for* $0 \leq i < j \leq m-1$*;*

iii) $a_i = \min([a_i] \cap G)$*;*

iv) $G = \bigcup_{j=0}^{m-1}(a_j + m\mathbb{N})$*;*

v) $c = a_{m-1} - (m-1)$.

Proof. i) Observe that $a_j,\ a_{j+1} \in G \setminus \bigcup_{i=0}^{j-1}(a_i + m\mathbb{N})$, so $a_j < a_{j+1}$, therefore $a_0 = 0 < a_1 < \cdots < a_{m-1}$.
ii) $a_i \not\equiv a_j \bmod m$ for $0 \leq i < j \leq m-1$, since $a_j \notin a_i + m\mathbb{N}$.
iii) Let $z \in [a_i] \cap G$, then $z \in G \setminus \bigcup_{j=0}^{i-1}(a_j + m\mathbb{N})$, so $a_i \leq z$.
iv) Since $a_j \not\equiv a_i \bmod m$ with $j \neq i$, it follows that $\{m, a_1, \ldots, a_{m-1}\}$ is a complete residue system modulo m. Hence $G \setminus \bigcup_{j=0}^{m-1}(a_j + m\mathbb{N}) = \emptyset$.
v) Notice that $a_{m-1} - (m-1) - 1 = a_{m-1} - m \notin G$. On the other hand, for all $r \geq 1$. Dividing $a_{m-1} - m + r$ by m, we have $a_{m-1} - m + r = \lambda m + a_i$ for some $\lambda \in \mathbb{Z}$ and some $i = 0, \ldots, m-1$. Then $r = (\lambda + 1)m + (a_i - a_{m-1}) \geq 1$, so $\lambda \geq 0$. Therefore, $a_{m-1} - m + r = \lambda m + a_i \in G$, for all $r \geq 1$. Thus $c = a_{m-1} - m + 1$. □

According to Proposition 2.13 (iii) and (iv) the elements of G are of the form $a_i + \lambda m$ for some $i = 0, \ldots, m-1$ and $\lambda \geq 0$, while the gaps of G are of the form $a_i + \lambda m$ for some $i = 0, \ldots, m-1$ and $\lambda < 0$.

The set

$$\mathcal{A} = \{a_0, \ldots, a_{m-1}\}$$

will be called the *Apéry set of G with respect to m.* When $m = n = \min(G \setminus \{0\})$, the set $\mathcal{A}$ will be called simply the *Apéry set* of G.

We could have defined the Apéry set with respect to m as

$$\{\alpha \in G;\ \alpha - m \notin G\}.$$

Indeed, if $\alpha \in \mathcal{A}$ then $\alpha \in G$ and $\alpha - m \notin G$, since, by Proposition 2.13, $\alpha = \min([\alpha] \cap G)$. This shows that $\mathcal{A} \subset \{\alpha \in G;\ \alpha - m \notin G\}$.

On the other hand, if $\alpha \in \{\alpha \in G;\ \alpha - m \notin G\}$,

$$\alpha = \lambda m + a_i,\ \text{for some } i = 0, \ldots, m-1,\ \lambda \geq 0$$

and

$$\alpha - m = \lambda m + a_i - m = (\lambda - 1)m + a_i \notin G,$$

which implies that $\lambda - 1 < 0$. So, $\lambda = 0$. Therefore $\alpha = a_i$, which shows that $\{\alpha \in G;\ \alpha - m \notin G\} \subset \mathcal{A}$.

Proposition 2.14. *Let* $G = \langle v_0, \ldots, v_g \rangle$*. The following assertions are equivalent.*

i) *The sequence* $v_0, \ldots, v_g$ *is nice;*

ii) *Every element* $m \in G$ *is uniquely representable in the form*

$$m = \sum_{i=0}^{g} s_i v_i \; 0 \leq s_i < n - i, \; i = 1, \ldots, g, \; s_0 \in \mathbb{N};$$

iii) *The Apéry set of G is given by*

$$\mathcal{A} = \left\{ \sum_{i=1}^{g} s_i v_i; \; 0 \leq s_i < n_i, \; i = 1, \ldots, g \right\};$$

iv) *The conductor c of G is given by*

$$c = \sum_{i=1}^{g} (n_i - 1) v_i - v_0 + 1.$$

Proof. (i) $\Rightarrow$ (ii) was proved in Proposition 2.9.
(ii) $\Rightarrow$ (iii): By the uniqueness of the representation of the elements in G, each number $\sum_{i=1}^{g} s_i v_i$ is minimal in its congruence class modulo $n = v_0$, so we have

$$\left\{ \sum_{i=1}^{g} s_i v_i; \; 0 \leq s_i < n_i \; i = 1, \ldots, g \right\} \subset \mathcal{A}.$$

Now, since

$$\sharp \left(\left\{ \sum_{i=1}^{g} s_i v_i; \; 0 \leq s_i < n_i, \; i = 1, \ldots, g \right\} \right) = n_1 n_2 \cdots n_g = n = \sharp \mathcal{A},$$

the result follows.
iii) $\Rightarrow$ iv): The largest element in $\mathcal{A}$ is $\sum_{i=1}^{g} (n_i - 1) v_i$, hence

$$c = a_{v_0 - 1} - v_0 + 1 = \sum_{i=1}^{g} (n_i - 1) v_i - v_0 + 1.$$

iv) $\Rightarrow$ (i): We will prove it by induction on g. For $g = 1$, we have $\langle v_0, v_1 \rangle$, $e_0 = v_0$, $e_1 = 1$, $n_0 = 1$, $n_1 = v_0$, so $n_1 v_1 = v_0 v_1 \in \langle v_0 \rangle$.

Suppose now that the result is true for $g - 1$. Let $v_0, \ldots, v_g$ be a sequence of coprime elements with associated integers n_i, e_i, $i = 0, \ldots, g$, such that the conductor c of $G = \langle v_0, \ldots, v_g \rangle$ is given by $c = \sum_{i=1}^{g} (n_i - 1) v_i - v_0 + 1$.

Now, consider the semigroup $G' = \langle v'_0, \dots, v'_{g-1} \rangle$, where $v'_i = \frac{v_i}{e_{g-1}}$ with $i = 0, \dots, g-1$. Its associated integers are $e'_i = \frac{e_i}{e_{g-1}}$ and $n'_i = n_i$.

From Proposition 2.7, we know that the conductor c' of G' satisfies

$$c' \leq \sum_{i=1}^{g-1} (n'_i - 1)v'_i - v'_0 + 1.$$

If the strict inequality holds, we would have $\sum_{i=1}^{g-1} (n'_i - 1)v'_i - v'_0 \in G'$, hence

$$\sum_{i=1}^{g-1} (n_i - 1)v_i - v_0 = e_{g-1} \left(\sum_{i=1}^{g-1} (n'_i - 1)v'_i - v'_0 \right) \in G,$$

which implies that

$$c - 1 = \sum_{i=1}^{g} (n_i - 1)v_i - v_0 = \sum_{i=1}^{g-1} (n_i - 1)v_i - v_0 + (n_g - 1)v_g \in G,$$

a contradiction. Then

$$c' = \sum_{i=1}^{g-1} (n'_i - 1)v'_i - v'_0 + 1,$$

and from the inductive assumtion one has that $v'_0, \dots, v'_{g-1}$ is nice sequence, so $n'_i v'_i \in \langle v'_0, \dots, v'_{i-1} \rangle$ for all $i = 1, \dots, g-1$. This means that

$$n_i v_i = e_{g-1} n'_i v'_i \in e_{g-1} \langle v'_0, \dots, v'_{i-1} \rangle = \langle v_0, \dots, v_{i-1} \rangle.$$

To finish the proof we only have to show that $n_g v_g \in \langle v_0, \dots, v_{g-1} \rangle$, which is equivalent to prove that $v_g \in G'$, since $n_g = e_{g-1}$. If $v_g \notin G'$, since G' is symmetric, it would follows that $c' - 1 - v_g \in G'$, so $e_{g-1}(c' - 1 - v_g) \in G$, hence

$$\begin{aligned} c - 1 &= e_{g-1} \left(\sum_{i=1}^{g-1} (n_i - 1)\frac{v_i}{e_{g-1}} - \frac{v_0}{e_{g-1}} \right) + e_{g-1}v_g - e_{g-1}v_g \\ &= e_{g-1} \left(\sum_{i=1}^{g-1} (n'_i - 1)v'_i - v'_0 - v_g \right) + e_{g-1}v_g \\ &= e_{g-1}(c' - 1 - v_g) + e_{g-1}v_g \in G, \end{aligned}$$

which is a contradiction. □

Corollary 2.15. *Let $0 < v_0 < v_1 < \cdots < v_g$ be a nice sequence of coprime integers such that $v_i \notin \langle v_0, \dots v_{i-1} \rangle$, for $i = 1, \dots, g$, then $v_0, \dots, v_g$ is a minimal system of generators of G.*

Proof. Let $x_0, \dots, x_{g'}$ be the minimal system of generators of G, then $v_0 = x_0$. Now we proceed by induction on i. Suppose that $v_j = x_j$ for $j = 0, \dots, i-1$. From the hypothesis we know that $x_i \in G \setminus \langle v_0, \dots, v_{i-1} \rangle$. To show that $x_i = v_i$ it is enough to prove that $v_i \leq z$ for all $z \in G \setminus \langle v_0, \dots, v_{i-1} \rangle$. Since the sequence $v_0, \dots, v_g$ is nice, by the unique representation we may write $z = \sum_{j=0}^{g} \lambda_j v_j$ with all λ_j nonnegative. Since $z \notin \langle v_0, \dots, v_{i-1} \rangle$, then one of the λ_j for $j = i, \dots, g$ is positive, so

$$z = \sum_{j=0}^{i-1} \lambda_j x_j + \sum_{j=i}^{g} \lambda_j v_j \geq v_i.$$

□

2.4 Strongly increasing semigroups

In this section we will study semigroups with an additional property given in the definition below.

A semigroup G with Apéry sequence $\{a_0, \dots, a_{n-1}\}$, where $n = \min(G \setminus \{0\})$, that satisfies the condition:

$$a_i + a_j \leq a_{i+j}, \text{ for all } 0 \leq i,\ j,\ i+j \leq n-1, \tag{2.1}$$

is called a *strongly increasing semigroup*.

From (2.1) we have $a_i \geq a_1 + a_{i-1}$, hence one has

$$a_i \geq i a_1, \text{ for all } i = 0, \dots, n-1 \tag{2.2}$$

Proposition 2.16. *If G is a strongly increasing semigroup of multiplicity n, then its Apéry sequence satisfies the following equality*

$$a_i + a_{n-1-i} = a_{n-1},\ \forall i = 0, \dots, n-1.$$

Proof. We will prove the equality by induction on i. For $i = 0$, the equality is trivially satisfied. Suppose that the proposition is verified for $0 \leq i \leq k-1$.

Since the Apéry sequence is a complete residue system modulo n, we have

$$a_{n-1} - a_k \equiv a_j \bmod n, \text{ for some } 0 \leq j \leq n-1,$$

so,

$$a_{n-1} = a_k + a_j + \lambda n, \text{ for some } \lambda \in \mathbb{Z}.$$

We must prove that $\lambda \leq 0$ and $j = n-1-k$.

If $\lambda > 0$ then $a_{n-1} - n = a_k + a_j + (\lambda - 1)n \in G$, which is a contradiction since $a_{n-1} - n$ is a gap of G, then

$$a_{n-1} = a_k + a_j + \lambda n \leq a_{k+j} + \lambda n \leq a_{k+j}.$$

Thus, $n - 1 \leq k + j \Rightarrow n - 1 - k \leq j \leq n - 1$, so there exists $0 \leq r \leq k - 1$ such that $j = n - 1 - k - r$. From the inductive assumption, one has

$$a_r + a_{n-1-r} = a_{n-1} \leq a_{k+n-1-k-r} = a_{n-1-r},$$

then $a_r = 0$, so $r = 0$. This means that $a_{n-1} = a_k + a_{n-1-k} + \lambda n \leq a_k + a_{n-1-k}$ with $\lambda \leq 0$.

Since G is a strongly increasing semigroup, we get $a_k + a_{n-1-k} \leq a_{n-1}$, so $a_{n-1} = a_k + a_{n-1-k}$. □

Proposition 2.17. *A Semigroup G is symmetric if and only if its Apéry sequence with respect to m, for some $m \in G \setminus \{0\}$, satisfies the equality*

$$a_i + a_{m-1-i} = a_{m-1}, \quad \textit{for all } i = 0, \ldots, m - 1.$$

Proof. Suppose first that G is symmetric. Fix an index i such that $0 \leq i \leq m - 1$, and consider the integer $a_{m-1} - m - a_i$, which is a gap of G, since otherwise, $c - 1 = a_{m-1} - m$ would be an element of G. Then $a_{m-1} - m - a_i = a_{j_i} + \lambda m$ for some $j_i = 0, \ldots, m - 1$ and $\lambda < 0$, so $a_{m-1} = a_i + a_{j_i} + (\lambda + 1)m$. We have to prove that $\lambda = -1$ and $j = m - 1 - i$.

If $\lambda < -1$, then

$$c - 1 - (a_i - m) = a_{m-1} - m - (a_i - m) = a_{j_i} + (\lambda + 1)m \notin G.$$

Since G is symmetric, this would imply that $a_i - m \in G$, which is a contradiction.

We have shown that for every $i = 0, \ldots, m - 1$ there exists $j_i = 0, \ldots, m - 1$ such that $a_{m-1} = a_i + a_{j_i}$. Now, because the Apéry sequence is increasing $a_{m-1} = a_{m-1} - a_0 > a_{m-1} - a_1 > \cdots > a_{m-1} - a_{m-1} = a_0$, so $j_0 > j_1 > \cdots > j_i > \cdots > j_{m-1}$, which implies that $j_i = m - 1 - i$.

Suppose now that the equality is true. We know that the conductor is $c = a_{m-1} - m + 1 \in G$. Then the condition $c - 1 - z \notin G$ is equivalent to $c - 1 - z = a_j + \lambda m$, for some $j = 0, \ldots, m - 1$ and λ a negative integer. This, in turn, is equivalent to

$$z = a_{m-1} - a_j - (\lambda + 1)m = a_{m-1-j} - (\lambda + 1)m \in G.$$

□

Corollary 2.18. *Every Strongly Increasing Semigroup is symmetric.*

Proof. If G is a strongly increasing semigroup, then from Proposition 2.16 its Apéry sequence satisfies

$$a_{n-1} = a_i + a_{n-1-i}, \ \forall i = 0, \ldots, n-1.$$

Now, from Proposition 2.17, we may conclude that G is symmetric. $\square$

If G is a semigroup with conductor, we define the following sequence of numbers:

$$x_0 = \text{mult}(G) \quad \text{and} \quad x_l = \min(G \setminus \gcd(x_0, \ldots, x_{l-1})\mathbb{N}), \ \text{for } l \geq 1. \tag{2.3}$$

Since $\gcd(G) = 1$, because G has a conductor, this process must stop at some point. Such finite sequence of numbers is called the *sequence in G where the* gcd *varies*.

LEMMA 2.19. *If G is any semigroup with conductor and $x_0, \ldots, x_l$, is the sequence in G where the* gcd *varies, then*

$$x_i \leq a_{n/e_{i-1}}, \textit{ for } i = 1, \ldots, l,$$

where $e_i = \gcd(x_0, \ldots, x_i)$.

Proof. Suppose that for some $i = 1, \ldots, l$ one has

$$x_i > a_{n/e_{i-1}},$$

so $n < a_1 < \cdots < a_{n/e_{i-1}} < x_i$. From the definition of the x_i, we have that e_{i-1} divides each of the elements $a_1, a_2, \ldots, a_{n/e_{i-1}}$. By euclidean division, we have that $a_k = l_k n + m_k e_{i-1}$ for unique $l_k \in \mathbb{N}$ and $0 < m_k < \frac{n}{e_{i-1}}$, for $k = 1, \ldots, \frac{n}{e_{i-1}}$. From the fact that $a_k \not\equiv a_{k'} \mod n$, if $k \neq k'$, it follows that $m_k \neq m_{k'}$ for $k, k' \in \{1, 2, \ldots, \frac{n}{e_{i-1}}\}$ with $k \neq k'$. So there are $\frac{n}{e_{i-1}}$ distinct elements m_k in the set $\{1, 2, \ldots, \frac{n}{e_{i-1}} - 1\}$, which is a contradiction. $\square$

THEOREM 2.20. *Let G be a strongly increasing semigroup and let $x_0, \ldots, x_l$ be the sequence of G where the* gcd *varies and put $x_0 = n$. If $H = \langle x_0, \ldots, x_l \rangle$, then*

i) $x_i = a_{n/e_{i-1}}$, *for* $i = 1, \ldots, l$;

ii) $x_0, \ldots, x_l$ *is a nice sequence and* $G = H$;

iii) $x_0, \ldots, x_l$ *is the minimal set of generators of* G;

iv) $x_i > n_{i-1} x_{i-1}$, *for all* $i = 1, \ldots, l$.

Proof. i) We know that $x_0 = n = \min(G)$, $x_1 = a_1$ and $n_1 = \frac{n}{e_1}$. Let c and c' be the conductors of G and H, respectively. From Proposition 2.7 one has

$$c' \leq \sum_{i=1}^{l} (n_i - 1)x_i - x_0 + 1.$$

Since $H \subset G$ we have $c \leq c'$, so

$$a_n - n = c - 1 \leq c' - 1 \leq \sum_{i=1}^{l} (n_i - 1)x_i - n =$$

$$\left(\frac{n}{e_i} - 1\right) x_1 + \sum_{i=2}^{l} (n_i - 1)x_i - n. \tag{2.4}$$

Since the semigroup is strongly increasing, from equality (2.2) and inequality (2.4), one has

$$a_{n-1} < a_n \leq \left(\frac{n}{e_1} - 1\right) a_1 + \sum_{i=2}^{l} (n_i - 1)x_i \leq$$

$$a_{n/e_1 - 1} + \sum_{i=2}^{l} (n_i - 1)x_i.$$

Therefore from Lemma 2.19 and from the fact that G is strongly increasing, we get

$$a_{n-1} < \sum_{i=2}^{l} (n_i - 1)x_i + a_{n/e_1 - 1} \leq \sum_{i=2}^{l} (n_i - 1)a_{n/e_{i-1}} + a_{n/e_1 - 1} \leq$$

$$\sum_{i=2}^{l} (n_i - 1)a_{(n_i - 1)n/e_{i-1}} + a_{n/e_1 - 1} \leq a_{\sum_{i=2}^{l}(n_i - 1)n/e_{i-1}} + a_{n/e_1 - 1}. \tag{2.5}$$

From Proposition 2.17 we have that

$$a_{n-1} = a_{n - n/e_1} + a_{n/e_1 - 1},$$

which implies that the inequalities in (2.5) are all equalities, hence

$$\sum_{i=2}^{l} (n_i - 1)x_i = \sum_{i=2}^{l} (n_i - 1)a_{n/e_{i-1}},$$

which in view of Lemma 2.19, implies that $x_i = a_{n/e_{i-1}}$, for $i = 2, \ldots, l$, so we are done.

ii) From all equalities established above and in view of (2.4) we have that

$$c = c' = \sum_{i=1}^{l}(n_i - 1)x_i - x_0 + 1.$$

From Proposition 2.14, $x_0, \dots, x_l$ is a nice sequence. And from Proposition 2.12 H is symmetric. Using Corollary 2.18 and from Remark 2.11, we conclude that $H = G$.
iii) From (ii) we know that $x_0, \dots, x_l$ is a nice sequence of coprime integers such that $x_i \notin \langle x_0, \dots, x_{i-1}\rangle$, for $i = 1, \dots, l$. By Corollary 2.15, $x_0, \dots, x_l$ is the minimal system of generators of G.
iv) Since G is strongly increasing, from equality (2.2), one has $a_i \geq ia_1$, $\forall\, i = 0, \dots, n-1$, hence

$$x_j = a_{\frac{n}{e_{j-1}}} \geq \frac{e_{j-2}}{e_{j-1}} a_{\frac{n}{e_{j-2}}} = n_{j-1}x_{j-1}, \quad \text{for} \quad j = 1, \dots, \ell.$$

To conclude the proof, just observe that equality in the above inequality does not hold, because of the variance of the gcd. □

Let $x, y \in \mathbb{N}^r$. We will say that x is *smaller* than y in the *reverse lexicographical order*, writing $x \preceq y$, if the last non-zero coordinate of $y - x$ is positive. This establishes a total order relation on $\mathbb{N}^r$.

Let $x_0, \dots, x_r$ be a sequence of positive relatively prime integers. Consider the integers $n_0, \dots, n_r$ associated to $x_0, \dots, x_r$ and define the set

$$E(x_0, \dots, x_r) = \{(s_1, \dots, s_r) \in \mathbb{N}^r;\ 0 \leq s_i < n_i\ i = 1, \dots, r\}.$$

We put on this set the reverse lexicographical order $\preceq$, and consider $\mathbb{N}$ with its natural order $\leq$.

Lemma 2.21. *Let $x_0, \dots, x_r$ be a sequence of positive relatively prime integers. The map*

$$\begin{array}{rcl} \lambda\colon E(x_0, \dots, x_r) & \to & \{0,\ 1, \dots, x_0 - 1\} \\ (s_1, \dots, s_r) & \mapsto & \sum_{i=1}^{r} s_i n_0 \dots n_{i-1} \end{array}$$

is an order preserving bijection.

Proof. The inequalities, for $j = 1, \dots, r$,

$$\sum_{i=1}^{j} s_i n_0 \dots n_{i-1} \leq (n_1 - 1)n_0 + (n_2 - 1)n_0 n_1 + \cdots + (n_j - 1)n_0 \dots n_{j-1} =$$

$$n_0 n_1 \dots n_j - n_0 < n_0 n_1 \dots n_j < x_0,$$

show that λ preserves orders and that its image is contained in $\{0, 1, \ldots, x_0 - 1\}$. Since λ is order preserving, it follows that it is injective and because both sets $E(x_0, \ldots, x_r)$ and $\{0, 1, \ldots, x_0 - 1\}$ have the same cardinality x_0, it follows that λ is a bijection. □

Lemma 2.22. *Let G be a strongly increasing semigroup of multiplicity n and let $x_0, \ldots, x_r$ be the sequence of G where the* gcd *varies. Then the map*

$$\begin{array}{rcl} \rho\colon E(x_0, \ldots, x_r) & \to & \mathcal{A} \\ (s_1, \ldots, s_r) & \mapsto & \sum_{i=1}^{r} s_i x_i \end{array}$$

is an order preseving bijection.

Proof. From Theorem 2.20 we know that $x_0, \ldots, x_r$ is a nice sequence, so from Proposition 2.14 it follows that

$$\mathcal{A} = \left\{ \sum_{i=1}^{r} s_i x_i;\ 0 \le s_i < n_i\ i = 1, \ldots, r \right\}.$$

Let $s = (s_1, \ldots, s_r),\ t = (t_1, \ldots, t_r) \in E(x_0, \ldots, x_r)$ be such that $s \preceq t$. Suppose by reduction to absurdity that $\rho(t) < \rho(s)$. Thus, $\sum_{i=1}^{r} t_i x_i < \sum_{i=1}^{r} s_i x_i$ and, consequently,

$$0 > \sum_{i=1}^{r} (t_i - s_i) x_i > \sum_{i=1}^{r} (t_i - s_i) n_{i-1} x_{i-1} > \sum_{i=1}^{r} (t_i - s_i) n_{i-1} n_{i-2} x_{i-2} >$$
$$\cdots > \sum_{i=1}^{r} (t_i - s_i) n_{i-1} n_{i-2} \cdots n_0 x_0 = x_0 \left(\sum_{i=1}^{r} (t_i - s_i) n_{i-1} n_{i-2} \cdots n_0 \right).$$

Therefore

$$\sum_{i=1}^{r} t_i n_{i-1} n_{i-2} \cdots n_0 < \sum_{i=1}^{r} s_i n_{i-1} n_{i-2} \cdots n_0,$$

hence $t \preceq s$, which is a contradiction, because λ in Lemma 2.21 is order preserving.

This shows that ρ preserves order. So, ρ is injective. Since $\sharp E = \sharp \mathcal{A} = x_0$, it follows that ρ is also surjective. □

Theorem 2.23. *Let G be a strongly increasing semigroup of multiplicity n and let $x_0, \ldots, x_r$ be the sequence of G where the* gcd *varies, then*

$$\sum_{i=1}^{r} s_i x_i = a_{\sum_{i=0}^{r} s_i n_0 \cdots n_{i-1}}.$$

Proof. This follows from the fact that $\mathcal{A} = \{\sum_{i=1}^{r} s_i x_i : \ 0 \leq s_i < n_i\}$ and that λ and ρ are order preserving bijections. $\square$

REMARK 2.24. *Let $0 < v_0 < \cdots < v_g$ be a minimal system of generators of a strongly increasing semigroup G. Then we have for any integer k, with $1 \leq k \leq g$,*

i) $(n_1 - 1)v_1 + \cdots + (n_k - 1)v_k \not\equiv 0 \bmod e_{k-1}$;

ii) $(n_1 - 1)v_1 + \cdots + (n_k - 1)v_k < v_{k+1}$;

iii) *If $(n_1 - 1)v_1 + \cdots + (n_k - 1)v_k = l_0 v_0 + l_1 v_1 + \cdots + l_k v_k$, with integers $l_0,\ l_k \geq 0$ and $0 \leq l_i < n_i$, for $i \in \{1, \ldots, k-1\}$, then $l_0 = 0$ and $l_i = n_i - 1$, for $i \in \{1, \ldots, k-1\}$;*

iv) *If $1 < j \leq k \leq g$, then $e_{j-1} v_j < e_{k-1} v_k$.*

Let us prove these assertions.
(i) Suppose that $(n_1 - 1)v_1 + \cdots + (n_k - 1)v_k \equiv 0 \bmod e_{k-1}$, then $(n_k - 1)v_k \equiv 0 \bmod e_{k-1}$, since $e_{k-1} = \gcd(v_0, \ldots, v_{k-1})$. Thus, $(n_k - 1)\frac{v_k}{e_k} \equiv 0 \bmod \frac{e_{k-1}}{e_k}$, which is a contradiction, since $\gcd\left(\frac{v_k}{e_k}, \frac{e_{k-1}}{e_k}\right) = 1$ and $0 < n_k - 1 < n_k = \frac{e_{k-1}}{e_k}$.
(ii) Follows easily by induction using the inequalities $v_i > n_{i-1} v_{i-1}$, $i = 1, \ldots, g$.
(iii) From (ii) we get that $l_k < n_k - 1$. Now, result follows from the uniqueness of the writing of the elements of G.
(iv) from Theorem 2.20 we know that, for all $i = 1, \ldots, g$ we have

$$v_i > n_{i-1} v_{i-1} = \frac{e_{i-2}}{e_{i-1}} v_{i-1},$$

so $e_{j-1} v_j < e_{k-1} v_k$ for $1 \leq j < k \leq g$, from which the result follows.

Chapter 3

Semigroup of Values of a Plane Branch

Recall that the semigroup of values of a branch (f) was defined as

$$G_f = \{I(f, g);\ g \in K[[X, Y]] \setminus \langle f \rangle\}.$$

We showed that this semigroup has a conductor. In this chapter we will show that it is strongly increasing, hence symmetric (cf. Corollary 2.18). For doing this, we will use the notion of Apéry polynomials.

3.1 The Apéry polynomials

The elements of the Apéry sequence of a branch (f) are elements of the semigroup G_f of (f). In this section we will construct some special elements in $K[[X, Y]]$ whose values in G_f are precisely the elements of the Apéry sequence.

Let $f \in K[[X, Y]]$ be an irreducible power series of multiplicity n. After a change of coordinates in $K[[X, Y]]$ we may suppose that f is regular in Y of order n and that n does not divide $m = I(f, Y)$. Recall that, at the end of Section 1.2, we observed that

$$\mathcal{O}_f = K[[X]] \oplus K[[X]]y \oplus \cdots \oplus K[[X]]y^{n-1}.$$

Let us define $M_{-1} = \{0\}$ and, for $k = 0, 1, \ldots, n-1$,

$$M_k = K[[X]] \oplus K[[X]]y \oplus \cdots \oplus K[[X]]y^k.$$

So,

$$K[[X]] = M_0 \subset M_1 \subset \cdots \subset M_{n-1} = \mathcal{O}_f.$$

Recall that $I(f, g) = \nu_f(g)$, where ν_f is the valuation introduced at the end of Chapter 1. In this section we will denote ν_f by ν. The following result, taken from [Ap], concerning the spaces M_k will play an important role in this theory.

Theorem 3.1 (Apéry). *Let $f \in K[[X,Y]]$ of multiplicity n, irreducible, regular of order n in Y and such that n does not divide $m = \mathrm{I}(f,Y)$. Then for every $k = 0, 1 \ldots, n-1$, there exists an element $y_k \in y^k + M_{k-1}$ such that $\nu(y_k) \notin \nu(M_{k-1})$.*

Proof. For $k = 0$, if we put $y_0 = 1$, we have $1 = y_0 \in y^0 + M_{-1}$ with $\nu(y_0) = 0 \notin \nu(M_{-1}) = \{\infty\}$.

Let $k < n$ be given. If $\nu(y^k) \notin \nu(M_{k-1})$, take $y_k = y^k$ and we are done. If $\nu(y^k) \in \nu(M_{k-1})$ then there exists $\phi_1 \in M_{k-1}$ such that $\nu(\phi_1) = \nu(y^k)$ hence there exist $c_1 \in K$ such that

$$\nu(y^k - c_1\phi_1) > \nu(y^k).$$

If $\nu(y^k - c_1\phi_1) \notin \nu(M_{k-1})$ we are done since we may take $y_k = y^k - c_1\phi_1$. But, if $\nu(y^k - c_1\phi_1) \in \nu(M_{k-1})$, there exists $c_2 \in K$ and $\phi_2 \in M_{k-1}$ such that

$$\nu(y^k - c_1\phi_1 - c_2\phi_2) > \nu(y^k - c_1\phi_1),$$

and so on. At some point this procedure will necessarily stop, because otherwise we would have

$$\nu\left(y^k - \sum_{i=1}^{\infty} c_i\phi_i\right) = \infty,$$

with $\sum_{i=1}^{\infty} c_i\phi_i \in M_{k-1}$, hence, from the Division Theorem (Chapter 1), $y^k - \sum_{i=1}^{\infty} c_i\phi_i = 0$ so $y^k \in M_{k-1}$, which is a contradiction. □

Observe that since $y_k \in y^k + M_{k-1}$, it follows that

$$O_f = K[[X]] + K[[X]]y_1 + \cdots + K[[X]]y_{n-1}.$$

Proposition 3.2 (Azevedo [Az]). *Suppose that for $k = 0, \ldots, , n-1$ we have elements $y_k \in y^k + M_{k-1}$ with $\nu(y_k) \notin \nu(M_{k-1})$, where $y_0 = 1$. Then for all i, j, with $0 \le i$, $j \le n-1$ and $i + j \le n-1$, we have*

$$\nu(y_i) + \nu(y_j) \le \nu(y_{i+j}).$$

Proof. We can write $y_i = y^i + a$ and $y_j = y^j + b$ with $a_i \in M_{i-1}$ and $b \in M_{j-1}$, then we have

$$y_iy_j = y^iy^j + ay^j + by^i + ab = y^{j+i} + c,$$

where $c = ay^j + by^i + ab \in M_{i+j-1}$.

We know that $y_{i+j} = y^{i+j} + d$ for some $d \in M_{i+j-1}$, so $y_iy_j = c - d + y_{i+j}$. If we put $e = c - d \in M_{i+j-1}$, it follows that $\nu(y_iy_j) = \nu(e + y_{i+j}) \ge \inf\{\nu(y_{j+i}), \nu(e)\}$. Since $\nu(y_{j+i}) \notin \nu(M_{i+j-1})$ and $\nu(e) \in \nu(M_{i+j-1})$, then $\nu(y_{j+i}) \neq \nu(e)$. So, $\nu(y_iy_j) = \inf\{\nu(y_{j+i}), \nu(e)\}$, implying that

$$\nu(y_i) + \nu(y_j) \le \nu(y_{i+j}).$$

□

Remark 3.3. *We have that* $\nu(y_j) > \nu(y_i)$*, whenever* $0 \le i < j \le n-1$.

Indeed, since for $l \ge 1$, $\nu(y_l) \notin \nu(M_{l-1})$, we have that $\nu(y_l) \neq 0$, hence it follows that $\nu(y_j) \ge \nu(y_{j-i}) + \nu(y_i) > \nu(y_i)$.

Proposition 3.4. *Let* $f \in K[[X, Y]]$ *be irreducible and regular in* Y *of order* $n = \mathrm{mult}(f)$*. Put* $y_0 = 1$ *and let* $y_1,\ y_2, \ldots, y_{n-1}$ *be elements of* O_f *such that* $y_k \in y^k + M_{k-1}$ *and* $\nu(y_k) \notin \nu(M_{k-1})$*. Denoting by* $[r]$ *the residual class of the integer* r*, modulo* n*, for all* $k = 0, \ldots, n-1$*, we have that*

i) $\nu(M_k) = \bigcup_{i=0}^{k}(\nu(y_i) + n\mathbb{N})$*;*

ii) $\nu(y_i) \not\equiv \nu(y_j) \bmod n$ *for all* $i,\ j = 0, \ldots, n-1$ *with* $i \neq j$*;*

iii) $\nu(y_k) = \min([\nu(y_k)] \cap G_f)$ *for all* $k = 0, \ldots, n-1$.

Proof. If $n = 1$ we have $G_f = \mathbb{N}$ and in this case the assertions (i)-(iii) are trivially satisfied. We may assume that $n > 1$.
i) By induction on k. For $k = 0$ we have $y_0 = 1$, then $\nu(y_0) = 0$ and $\nu(M_0) = \nu(K[[X]]) = n\mathbb{N}$. Now, suppose that for some k such that $1 \le k \le n-1$ we have

$$\nu(M_{k-1}) = \{\nu(y_i) + \lambda n;\ 0 \le i \le k-1,\ \lambda \ge 0\}.$$

We know that $M_k = M_{k-1} + K[[X]]y_k$, so for any element $\beta \in M_k$, we may write $\beta = \alpha + a(X)y_k$ with $\alpha \in M_{k-1}$ and $a(X) \in K[[X]]$. It is enough to prove that $\nu(\alpha) \neq \nu(a(X)y_k)$. Indeed, if $\nu(\alpha) = \nu(a(X)y_k) = \nu(a(X)) + \nu(y_k)$, using the inductive hypothesis, we would have, for some $i \le k-1$,

$$\nu(y_i) + \lambda n = \nu(y_k) + \mu n.$$

From Remark 3.3, we have $\lambda > \mu$, then

$$\nu(y_k) = \nu(y_i) + (\lambda - \mu)n \in \nu(M_{i-1}) \subset \nu(M_{k-1}),$$

which is a contradiction.
ii) Suppose that for some integers $i,\ j \in \{0, \ldots, n-1\}$ with $i < j$ we have $\nu(y_i) \equiv \nu(y_j) \bmod n$, so for some positive integer λ (see Remark 3.3), $\nu(y_j) = \nu(y_i) + \lambda n$, from (i) we get $\nu(y_j) \in \nu(M_i) \subset \nu(M_{j-1})$, a contradiction.
iii) From (ii), we have that each residual class modulo n contains exactly one of the integers $\nu(y_k)$, $k = 0, \ldots, n-1$. On the other hand, we know that

$$\mathbb{N} \cap [\nu(y_k)] \cup G_f = \{\nu(y_k) + \lambda n :\ \lambda \ge 0\}.$$

Therefore,

$$\nu(y_k) = \min([\nu(y_k)] \cap G_f) \text{ for all } k = 0, \ldots, n-1.$$

□

Conditions (ii) and (iii) in Proposition 3.4 say that $\nu(y_0) < \cdots < \nu(y_{n-1})$ is the Apéry sequence of G_f. Also, Proposition 3.2 gives a very important property of the Apéry sequences: G_f is a strongly increasing semigroup.

Polynomials Y_i, $i = 0, \dots, n-1$, in $K[[X]][Y]$ of degree less than n whose residual classes mod f are the elements y_i, will be called *Apéry polynomials*. Notice that these polynomials are not unique.

Corollary 3.5. *Let i and j be two distinct integers such that $i,\ j = 0, \dots, n-1$ and let $\alpha_i(X),\ \alpha_j(X)$ in $K[[X]] \setminus \{0\}$. Then*

$$\nu(\alpha_i(X)y_i) \not\equiv \nu(\alpha_j(X)y_j) \bmod n.$$

Proof. We have already observed that $\nu(\alpha_i(X)) = \lambda_i n$ and $\nu(\alpha_j(X)) = \lambda_j n$, for some natural numbers λ_i and λ_j. Assuming $j > i$ and $\nu(\alpha_i(X)y_i) = \nu(\alpha_j(X)y_j)$, we get $\nu(y_i) - \nu(y_j) = (\lambda_j - \lambda_i)n$, so $\nu(y_i) \equiv \nu(y_j) \bmod n$, a contradiction according to Proposition 3.4. □

Corollary 3.6.

$$O_f = K[[X]] \oplus K[[X]]y_1 \oplus \cdots \oplus K[[X]]y_{n-1}.$$

Proof. We know that $O_f = K[[X]] + K[[X]]y_1 + \cdots + K[[X]]y_{n-1}$. It is sufficient to prove that $y_0, \dots, y_{n-1}$ are independent over $K[[X]]$. In fact, suppose that we had a non-trivial relation

$$\alpha_0(X)y_0 + \alpha_1(X)y_1 + \cdots + \alpha_{n-1}(X)y_{n-1} = 0.$$

From Corollary 3.5, we have $\nu(\alpha_i(X)y_i) \neq \nu(\alpha_j(X)y_j)$ for $i,\ j \in \{0, \dots, n-1\}$ and $i \neq j$. So, there exists $i \in \{0, \dots, n-1\}$ such that

$$\infty = \nu(0) = \nu(\alpha_0(X)y_0 + \alpha_1(X)y_1 + \cdots + \alpha_{n-1}y_{n-1}) = \nu(\alpha_i(X)y_i),$$

which is a contradiction. □

Corollary 3.7. *Let $1 = z_0,\ z_1, \dots, z_{n-1} \in O_f$ be such that*

a) $z_k \in y^k + M_{k-1}$ *for all $k = 0, \dots, n-1$; and*

b) $\nu(z_0), \dots, \nu(z_{n-1})$ *are pairwise non-congruent modulo n.*

Then $\nu(z_k) = \nu(y_k)$ for all $k = 0, \dots, n-1$.

Proof. From Proposition 3.4, $\nu(M_{k-1})$ intersects only k residual classes modulo n and $\nu(z_i) \in \nu(y^i + M_{i-1}) \subset \nu(M_{k-1})$ for all $i = 0, \dots, k-1$. It follows that $\nu(z_k) \notin \nu(M_{k-1})$ and by hypothesis $z_k \in y^k + M_{k-1}$. From Proposition 3.4 we have

$$\nu(z_k) = \min([\nu(z_k)] \cap G_f) = \min([\nu(y_k)] \cap G_f) = \nu(y_k).$$

□

3.2 Key-polynomials

In this section we will study properties of the Apéry polynomials Y_i attached to a given branch (f) defined by an irreducible series $f \in K[[X,Y]]$, regular in Y of order $n = \mathrm{mult}(f)$.

Since the residual class of Y_i is in $y^i + M_{i-1}$, we may assume that $Y_i = Y^i + A_{i,1}Y^{i-1} + \cdots + A_{i,i} \in K[[X]][Y]$. We have the following result:

Proposition 3.8. *Each polynomial Y_i is a Weierstrass polynomial.*

Proof. We want to prove that $\mathrm{mult}(A_{i,j}) > j$ for all i, j.

First suppose that for some i and some j we have that $\mathrm{mult}(A_{i,j}) < j$. This implies that $\mathrm{mult}(Y_i) < i$.

If the tangent cone of Y_i has a factor not of the form Y^r for some r, then we take a factor g of Y_i with tangent cone not containing Y as a factor. Suppose that $Y_i = gh$, then

$$a_i = \mathrm{I}(f, Y_i) = \mathrm{I}(f,g) + \mathrm{I}(f,h) = \mathrm{mult}(f)\,\mathrm{mult}(g) + \mathrm{I}(f,h) = n\,\mathrm{mult}(g) + \mathrm{I}(f,h).$$

This implies that $a_i \equiv \mathrm{I}(f,h) \bmod n$, which is a contradiction, since a_i is the least element of G_f in its residual class modulo n.

Then we have shown that the tangent cone of Y_i is Y^r for some $r < i$. Now, applying the Weierstrass Preparation Theorem, we may multiply Y_i by a unit u in order to get $uY_i = Y^r + B_1Y^{r-1} + \cdots + B_r \in K[[X]][Y]$, where $r < i$. Hence $a_i = \mathrm{I}(f, Y_i) = \mathrm{I}(f, uY_i) \in \nu(M_r) \subset \nu(M_{i-1})$, which is a contradiction.

So $\mathrm{mult}(A_{i,j}) \geq j$. If $\mathrm{mult}(A_{i,j}) = j$, then $\mathrm{mult}(Y_i) = i$ and the tangent cone of Y_i is not of the form Y^i. Applying the same argument as above we decompose $Y_i = gh$ such that the tangent cone of g does not contain Y as a factor and get, in the same way, a contradiction. □

Suppose that f is an irreducible power series of multiplicity n, let G_f be its semigroup of values and $v_0, \ldots, v_g$ be the minimal system of generators. After a change of coordinates in $K[[X,Y]]$,we may assume that f is regular in Y of order n.

Consider now Weierstrass polynomials $f_{-1} = X$ and $f_i = Y_{\frac{n}{e_i}}$, for $i = 0, \ldots, g-1$, so that

$$\deg_Y f_i = \frac{n}{e_{i-1}} \quad \text{and} \quad \nu_f(\bar{f_i}) = \mathrm{I}(f, f_{i-1}) = a_{n/e_i} = v_{i+1}. \tag{3.1}$$

Since $\nu_f(\bar{f_i}) = v_{i+1}$, $i = 0, \ldots, g-1$, cannot be written as the sum of two nonzero elements in G_f, then f_i is an irreducible power series.

Irreducible Weierstrass polynomials $f_{-1} = X, f_0, \ldots, f_{g-1}$ (not uniquely determined) satisfying (3.1) will be called *key-polynomials* of f.

Chapter 4

The Contact Among Branches

The contact among branches was used by M. Merle to prove in [Me] an important factorization theorem over the complex numbers, where in this case, the contact among branches is defined through Puiseux parametrizations, which we do not have in positive characteristic. Later, A. Granja in [Gr] generalized Merle's result extending it also for arbitrary characteristic, but using in this case Hamburger-Noether expansions which are the substitute for Puiseux parametrizations in any characteristic. The proof we give here is due to A. Ploski and E. García Barroso in [GB-P1], which uses the log-distance and is shorter and more elegant than the original proofs.

4.1 Log-distance

Let A be an nonempty set. A *log-distance* between elements in A is a function $d\colon A \times A \to \mathbb{R} \cup \{\infty\}$ such that, for all $a, b, c \in A$,

1. $d(a, a) = \infty$;
2. $d(a, b) = d(b, a)$;
3. $d(a, b) \geq \inf\{d(a, c),\ d(c, b)\}$.

The condition (3), called the *strong triangular inequality* (STI), is equivalent to the following one:

(3') At least two of the numbers $d(a, b)$, $d(a, c)$ and $d(b, c)$ are equal and the third is not smaller than them.

Let us prove this equivalence.

Suppose that (3') holds. Without loss of generality, we may assume that $d(a, c) = d(b, c)$, then $d(a, b)$ is not smaller than $d(a, c)$ and $d(b, c)$, therefore $d(a, b) \geq \inf\{d(a, c),\ d(c, b)\}$. This shows that (3) holds.

Conversely, let us suppose that (3) holds, then

$$d(a,b) \geq \inf\{d(a,c),\ d(c,b)\}, \tag{4.1}$$

$$d(a,c) \geq \inf\{d(a,b),\ d(c,b)\},$$

$$d(c,b) \geq \inf\{d(a,c),\ d(a,b)\}.$$

Because of the symmetry of the above relations, we may assume that $d(a,b) \leq d(b,c) \leq d(a,c)$. To prove (3'), we must prove that $d(a,b) = d(b,c)$. Suppose by reduction to absurdity that $d(a,b) < d(b,c)$. From (4.1), one has $d(a,c) < d(c,b)$, a contradiction.

Lemma 4.1. *Let d be a log-distance on a set A and let $a_1, \ldots, a_m, b_1, \ldots, b_n, c \in A$, then at least one of the following conditions holds:*

i) *There exists $j \in \{1, \ldots, n\}$ such that for any $i \in \{1, \ldots, m\}$, $d(a_i, c) \leq d(a_i, b_j)$.*

ii) *There exists $i \in \{1, \ldots, m\}$ such that for any $j \in \{1, \ldots, n\}$, $d(b_j, c) \leq d(a_i, b_j)$.*

Proof. Suppose that neither (i) nor (ii) holds. Then for any $j \in \{1, \ldots, n\}$ there exists and index $p(j) \in \{1, \ldots, m\}$ such that $d(a_{p(j)}, c) > d(a_{p(j)}, b_j)$ and for any $i \in \{1, \ldots, m\}$ there exists $s(i) \in \{1, \ldots, n\}$ such that $d(b_{s(i)}, c) > d(a_i, b_{s(i)})$. Applying the STI condition to $a_{p(j)}, b_j, c$ and to $a_i, b_{s(i)}, c$, we get

$$d(a_{p(j)}, c) > d(b_j, c) = d(a_{p(j)}, b_j), \tag{4.2}$$

and

$$d(b_{s(i)}, c) > d(a_i, c) = d(a_i, b_{s(i)}). \tag{4.3}$$

Without loss of generality, we may assume that

$$d(a_{p(1)}, b_1) = \max_{j \in \{1,\ldots,m\}} \{d(a_{p(j)}, b_j\}.$$

From (4.2) and (4.3) and again (4.2), we get

$$d(a_{p(1)}, b_1) < d(a_{p(1)}, c) < d(b_{s(p(1))}, c) = d(a_{p(s(p(1))}, b_{s(p(1))}),$$

a contradiction. □

Example 4.2. *The function $d : K[[s]] \times K[[s]] \to \mathbb{R} \cup \{\infty\}$ given by*

$$d(\alpha(s), \beta(s)) = \mathrm{mult}(\alpha(s) - \beta(s))$$

is a log-distance in $K[[s]]$.

Indeed, suppose that $\alpha(s)-\beta(s) = s^a u$, $\alpha(s)-\gamma(s) = s^b v$ and $\gamma(s)-\beta(s) = s^c w$, where u, v and w are units. Suppose that $a = \min(a,b,c)$. From the last two equalities above, we get that $s^a u = \alpha(s)-\beta(s) = s^b v + s^c w$. Since $a = \min(a,b,c)$, it follows that $a = b$ or $a = c$ and if $a = b$ and since $b \geq a$ and $c \geq a$, the result follows.

THEOREM 4.3. *Let* (l) *be a smooth branch. If* $d_l\colon K[[X,Y]]\times K[[X,Y]] \to \mathbb{R}\cup\{\infty\}$ *is given by*

$$d_l(f,g) = \frac{\mathrm{I}(f,g)}{\mathrm{I}(f,l)\,\mathrm{I}(g,l)}, \tag{4.4}$$

then d_l *is a log-distance in the set of all branches different from* (l).

Proof. Without loss of generality, we may assume after a change of coordinates in $K[[X,Y]]$ that $l = X$.

Because of the symmetry of the intersection index, we have that $d_X(f,g) = d_X(g,f)$. It is also clear that $d_X(f,f) = +\infty$.

It then suffices to check the STI condition. Let (f), (g), (h) be three branches different from (X), so they are all regular in Y. After multiplication by units, we may assume that they are distinguished polynomials in Y of degrees $m = \mathrm{I}(f,X)$, $n = \mathrm{I}(g,X)$ and $p = \mathrm{I}(h,X)$. Using Theorem 1.4, there exist power series $\alpha(s)$, $\alpha_i(s)$, $i = 1,\dots,m$, $\beta_j(s)$, $j = 1,\dots,n$ and $\gamma_k(s)$, $k = 1,\dots,p$, such that

$$\begin{aligned} f(\alpha(s),Y) &= \textstyle\prod_{i=1}^{m}(Y-\alpha_i(s)),\\ g(\alpha(s),Y) &= \textstyle\prod_{j=1}^{n}(Y-\beta_j(s)),\\ h(\alpha(s),Y) &= \textstyle\prod_{k=1}^{p}(Y-\gamma_k(s)). \end{aligned}$$

Let us consider the log-distance of Example 4.2 and apply Lemma 4.1 to $\alpha_1(s),\dots,\alpha_m(s)$, $\beta_1(s),\dots,\beta_n(s)$ and $\gamma(s) = \gamma_k(s)$, where $k \in \{1,\dots,p\}$ is fixed. Then

1) There exists $j \in \{1,\dots,n\}$ such that, for all $i \in \{1,\dots,m\}$,

$$\mathrm{mult}(\alpha_i(s)-\gamma(s)) \leq \mathrm{mult}(\alpha_i(s)-\beta_j(s)); \textit{ or}$$

2) There exists $i \in \{1,\dots,m\}$ such that, for all $j \in \{1,\dots,n\}$,

$$\mathrm{mult}(\beta_j(s)-\gamma(s)) \leq \mathrm{mult}(\alpha_i(s)-\beta_j(s)).$$

If (1) holds, then $\sum_{i=0}^{m}\mathrm{mult}(\alpha_i(s)-\gamma(s)) \leq \sum_{i=0}^{m}\mathrm{mult}(\alpha_i(s)-\beta_j(s))$; that is,

$$\mathrm{mult}(f(\alpha(s),\gamma(s))) \leq \mathrm{mult}(f(\alpha(s),\beta_j(s))).$$

Since $h(\alpha(s), \gamma(s)) = 0$ and $g(\alpha(s), \beta_j(s)) = 0$, by Theorem 1.5, we get

$$\frac{\mathrm{I}(h, f)}{\mathrm{I}(h, X)} \operatorname{mult}(\alpha(s)) = \operatorname{mult}(f(\alpha(s), \gamma(s))) \leq \operatorname{mult}(f(\alpha(s), \beta_j(s))) =$$

$$= \frac{\mathrm{I}(g, f)}{\mathrm{I}(g, X)} \operatorname{mult}(\alpha(s)),$$

therefore,

$$d_X(f, h) = \frac{\mathrm{I}(h, f)}{\mathrm{I}(h, X)\,\mathrm{I}(f, X)} \leq \frac{\mathrm{I}(g, f)}{\mathrm{I}(g, X)\,\mathrm{I}(f, X)} = d_X(f, g). \tag{4.5}$$

On the other hand, if (2) holds, in the same way, one shows that

$$d_X(g, h) = \frac{\mathrm{I}(h, g)}{\mathrm{I}(g, X)\,\mathrm{I}(g, X)} \leq \frac{\mathrm{I}(g, f)}{\mathrm{I}(g, X)\,\mathrm{I}(f, X)} = d_X(f, g) \tag{4.6}$$

Using (4.5) and (4.6) we have that, in any case, $d_X(g, f) \geq \inf\{d_X(g, h),\ d_X(f, h)\}$ □

Corollary 4.4. *The function*

$$d(f, g) = \frac{\mathrm{I}(f, g)}{\operatorname{mult}(f)\operatorname{mult}(g)}$$

is a log-distance in the set of all plane branches.

Proof. Given (f),(g) in the set of all branches, take (l) different from the tangents of (f) and (g), then

$$d_l(f, g) = \frac{\mathrm{I}(f, g)}{\mathrm{I}(f, l)\,\mathrm{I}(g, l)} = \frac{\mathrm{I}(f, g)}{\operatorname{mult}(f)\operatorname{mult}(g)} \tag{4.7}$$

is a logarithmic distance. □

We define the *relative contact index* of two branches (f) and (g) with respect to (l) as being the number $d_l(f, g)$. The number $d(f, g)$ will be called simply the *contact index* of f and g.

In general, $d(f, g)$ is a rational number greater or equal than 1. One has that $d(f, g) = 1$ if and only if f and g have distinct tangents. Notice that this notion of contact among branches is not the same as the classical one that measures the coincidence of the Puiseux parametrizations of the two branches up to a certain order.

Example 4.5. *Let (f) be a plane branch and let $f_0, \ldots, f_{g-1}$ be a sequence of key-polynomials for f. Then*

$$d(f, f_{i-1}) = \frac{e_{i-1} v_i}{n^2},\ i = 1, \ldots, g.$$

The numbers appearing on the right hand side of the above equality will have an important role in the next sections.

4.2 Branches with high contact

We are going now to study pairs of branches with relative high contact. We will show, in particular, that given two irreducible branches (f) and (h) such that $d(f,h)$ is sufficiently high, then the semigroups $G_f = \langle v_0, \dots, v_g \rangle$ and $G_h = \langle v'_0, \dots, v'_{g'} \rangle$, where $v_0 = n = \mathrm{I}(f, X)$ and $v'_0 = n' = \mathrm{I}(h, X)$, are closely related up to a certain order.

After a change of coordinates, if necessary, we may assume without loss of generality that f and g are regular in Y of order equal to their multiplicity. We will denote by $e_i,\ n_i$ (respectively $e'_i,\ n'_i$) the integers attached to G_f (respectively to G_h) and by $f_0, \dots, f_{g-1}$ key-polynomials of f (respectively $h_0, \dots, h_{g'-1}$ of h).

In the sequel we will need the following remark.

Remark 4.6. *If the equalities* $\frac{v_i}{n} = \frac{v'_i}{n'}$, *for all* $i \in \{1, \dots, k\}$, *where* $k \leq \min\{g, g'\}$, *hold, then* $\frac{n}{e_i} = \frac{n'}{e'_i}$ *for all* $i \in \{1, \dots, k\}$.

The proof of this assertion is by elementary arithmetic.

Lemma 4.7. *Let* $n = \mathrm{I}(f, X) > 1$, *and suppose that* $d_X(f, h) > \frac{e_{k-1} v_k}{n^2}$ *for some integer* $k \in \{1, \dots, g\}$. *Then*

i) $n\,\mathrm{I}(h, f_{i-1}) = n' v_i$, *for all* $i \in \{1, \dots, k\}$;

ii) $n' \equiv 0 \bmod \frac{n}{e_k}$;

iii) $d_X(f_{i-1}, h_{i-1}) = \frac{e_{i-1} e'_{i-1} \mathrm{I}(f_{i-1}, h_{i-1})}{nn'}$, *for* $0 < i \leq \min\{k, g'\}$;

iv) *We have* $n' > 1$ *and* $\frac{v_1}{n} = \frac{v'_1}{n'}$;

v) *Let* $0 < i < k$, $i < g'$ *and assume that* $\frac{v_j}{n} = \frac{v'_j}{n'}$, *for all* $j \leq i$. *Then* $i < g'$ *and* $\frac{v_{i+1}}{n} = \frac{v'_{i+1}}{n'}$.

Proof. (i) Fix $i \in \{1, \dots, k\}$. From our hypothesis, by Remark 2.24 (iv) and by Example 4.5, we have

$$d_X(f, h) = \frac{\mathrm{I}(f, h)}{nn'} > \frac{e_{k-1} v_k}{n^2} > \frac{e_{i-1} v_i}{n^2} = d_X(f, f_{i-1}).$$

From the STI condition applied to f, f_{i-1} and h, we have

$$\frac{\mathrm{I}(h, f_{i-1}) e_{i-1}}{n' n} = d_X(h, f_{i-1}) = d_X(f, f_{i-1}) = \frac{e_{i-1} v_i}{n^2},$$

from which the result follows.

ii) From part (i), we have

$$n'e_k = n' \gcd(v_i; i = 0, \ldots, k) = n \gcd(\mathrm{I}(h, f_{i-1}); i = 0, \ldots, k) \equiv 0 \bmod n.$$

iii) One has

$$d_X(f_{i-1}, h_{i-1}) = \frac{\mathrm{I}(f_{i-1}, h_{i-1})}{\mathrm{I}(f_{i-1}, X)\,\mathrm{I}(h_{i-1}, X)} = \frac{e_{i-1}e'_{i-1}\,\mathrm{I}(f_{i-1}, h_{i-1})}{nn'}.$$

iv) That $n' > 1$ follows from (ii). Now, Applying (i) and Example 4.5, one gets

$$d_X(h, f_0) = \frac{e_0 v_1}{n^2} = \frac{v_1}{n} \notin \mathbb{N} \quad \text{and} \quad d_X(h, h_0) = \frac{e'_0 v'_1}{(n')^2} = \frac{v'_1}{n'} \notin \mathbb{N}.$$

On the other hand,

$$d_X(f_0, h_0) = \frac{e_0 e'_0\,\mathrm{I}(f_0, h_0)}{nn'} = \mathrm{I}(f_0, h_0) \in \mathbb{N},$$

so, by the STI condition applied to h, h_0 and f_0, one gets $\frac{v_1}{n} = \frac{v'_1}{n'}$.

v) Since $\frac{v_j}{n} = \frac{v'_j}{n'}$, by Remark 4.6, $\frac{e_j}{n} = \frac{e'_j}{n'}$. By (ii) there exists an integer $l > 0$ such that $n' = l\frac{n}{e_k}$. Thus,

$$e'_i = n'\frac{e_i}{n} = l\frac{n}{e_k}\frac{e_i}{n} = l\frac{e_i}{e_k} > 1,$$

since $i < k$, then, obviously, $i < g'$. From what we obtained above, we have

$$d_X(h, f_i) = \frac{e_i v_{i+1}}{n^2}, \quad d_X(h, h_i) = \frac{e'_i v'_{i+1}}{(n')^2} \text{ and } d_X(f_i, h_i) = \frac{e_i e'_i\,\mathrm{I}(f_i, h_i)}{nn'}.$$

We claim that $d_X(h, f_i) \neq d_X(f_i, h_i)$, because if the equality was true, we would have

$$\frac{e_i v_{i+1}}{n^2} = \frac{e_i e'_i\,\mathrm{I}(f_i, h_i)}{nn'},$$

then

$$v_{i+1} = \frac{ne'_i\,\mathrm{I}(f_i, h_i)}{n'} = \frac{n'e_i\,\mathrm{I}(f_i, h_i)}{n'} = e_i\,\mathrm{I}(f_i, h_i),$$

which is absurd, since $v_{i+1} \not\equiv 0 \bmod e_i$.

We also claim that $d_X(h, h_i) \neq d_X(f_i, h_i)$, because otherwise $\frac{e'_i v'_{i+1}}{(n')^2} = \frac{e_i e'_i\,\mathrm{I}(f_i, h_i)}{nn'}$, thus

$$v'_{i+1} = \frac{n'e_i\,\mathrm{I}(f_i, h_i)}{n} = \frac{ne'_i\,\mathrm{I}(f_i, h_i)}{n} = e'_i\,\mathrm{I}(f_i, h_i),$$

which is absurd too, by the same reason.

Therefore, by the STI condition, $d_X(h, f_i) = d_X(h, h_i)$, hence $\frac{e_i v_{i+1}}{n^2} = \frac{e'_i v'_{i+1}}{(n')^2}$, so $\frac{v_{i+1}}{n} = \frac{v'_{i+1}}{n'}$. □

Theorem 4.8. *Let* $n = \mathrm{I}(f, X) > 1$ *and suppose that* $d_X(f, h) > \frac{e_{k-1}v_k}{n^2}$ *for an integer* $k \in \{1, \ldots, g\}$. *Then* $k \leq g'$ *and* $\frac{v_i}{n} = \frac{v'_i}{n'}$, *for all* $i \in \{1, \ldots, k\}$, *and the first* k *key-polynomials* $f_0, \ldots, f_{k-1}$ *of* f *are the first* k *key-polynomials of* h.

Proof. From Lemma 4.7, we conclude that $k \leq g'$ and $\frac{v_i}{n} = \frac{v'_i}{n'}$ for $i \in \{1, \ldots, k\}$, which proves the first part of the theorem. By Lemma 4.7 (i) we have $\mathrm{I}(h, f_{i-1}) = \frac{n' v_i}{n} = v'_i$ and since $\frac{v_i}{n} = \frac{v'_i}{n'}$, by Remark 4.6 we get $\deg(f_{i-1}) = \frac{n}{e_{i-1}} = \frac{n'}{e'_{i-1}}$. Therefore, f_{i-1} is a key-polynomial of h, for $i \in \{1, \ldots, k\}$. □

Theorem 4.9. *Let* (ϕ) *be an algebroid plane curve such that* $d_X(f, \phi) > \frac{e_{k-1}v_k}{n^2}$, *for some integer* k. *If* $n' = \mathrm{I}(\phi, X) = \frac{n}{e_k}$, *then* ϕ *is irreducible and* $G_\phi = \langle \frac{v_0}{e_k}, \ldots, \frac{v_k}{e_k} \rangle$.

Proof. First we are going to prove that ϕ is irreducible. By hypothesis we have $\mathrm{I}(f, \phi) > n_k v_k$ and suppose that $\phi = \phi_1 \cdots \phi_s$, where $\phi_1, \ldots, \phi_s \in K[[X, Y]]$ are irreducible. Let us show that there exists $j \in \{1, \ldots, h\}$ such that

$$\frac{\mathrm{I}(f, \phi_j)}{\mathrm{I}(X, \phi_j)} > \frac{e_{k-1}v_k}{n}. \tag{4.8}$$

If $\mathrm{I}(f, \phi_j) \leq \frac{e_{k-1}v_k}{n} \mathrm{I}(X, \phi_j)$ for all $j \in \{1, .., s\}$, then

$$\mathrm{I}(f, \phi) = \sum_{j=1}^{s} \mathrm{I}(f, \phi_j) \leq \sum_{j=1}^{s} \frac{e_{k-1}v_k}{n} \mathrm{I}(X, \phi_j) = \frac{e_{k-1}v_k}{n} \mathrm{I}(X, h) = \frac{e_{k-1}v_k}{n} \cdot \frac{n}{e_k} = n_k v_k,$$

which contradicts the assumption about $\mathrm{I}(f, \phi)$. By Lemma 4.7 (ii) the inequality (4.8) implies that $\mathrm{I}(X, \phi_j) = q\frac{n}{e_k}$, for some integer $q > 0$. On the other hand, $\mathrm{I}(X, \phi_j) \leq \mathrm{I}(X, h)$, so $q\frac{n}{e_k} \leq \frac{n}{e_k}$, then $q = 1$. Since ϕ_j is irreducible and divides ϕ, it follows that $\mathrm{ord}_Y \phi_j = \mathrm{ord}_Y \phi$, thus ϕ_j is associated with ϕ, therefore ϕ is irreducible.

Let $n' = v'_0, v'_1, \ldots, v'_{g'}$ the minimal set of generators of G_ϕ and let e'_i their associated integers. From Lemma 4.7 we know that $n' \equiv 0 \bmod \frac{n}{e_k}$. If $n' = \frac{n}{e_k}$, from Theorem 4.8 one has that $\frac{v_i}{n} = \frac{v'_i}{n'}$, for all $0 < i \leq k$. Then $\frac{v_i}{n} = \frac{v'_i}{n/e_k}$, so $v'_i = \frac{v_i}{e_k}$, for $i = 1, \ldots, k$. Since $\gcd\left(\frac{v_1}{e_k}, \ldots, \frac{v_k}{e_k}\right) = 1$, it follows that $G_\phi = \langle \frac{v_1}{e_k}, \ldots, \frac{v_k}{e_k} \rangle$. □

Corollary 4.10 (Abhyankar-Moh irreducibility criterion). *If* $\mathrm{I}(X, \phi) = n$ *and* $d_X(f, \phi) > \frac{e_{g-1}v_g}{n^2}$ *then* ϕ *is irreducible and* $G_f = G_\phi$.

Proof. The Corollary follows from Theorem 4.9, where we take $k = g$. □

Remark 4.11. *Under the notations and assumptions of Theorem 4.9, we get*

$$d_X(f, \phi) > \frac{e_{k-1}v_k}{n^2} = \frac{e_{k-1}}{n}\frac{v_k}{n} = \frac{e'_{k-1}}{n'}\frac{v'_k}{n'} = \frac{e'_{k-1}v'_k}{(n')^2}.$$

Corollary 4.12. *If (f) is a branch with semigroup of values $G_f = \langle v_0, \dots, v_g \rangle$ and f_{k-1} is a key-polynomial of f, then $G_{f_{k-1}} = \langle \frac{v_0}{e_{k-1}}, \dots, \frac{v_{k-1}}{e_{k-1}} \rangle$,*

Proof. Since $I(f_{k-1}, X) = \frac{n}{e_{k-1}}$ and

$$d(f, f_{k-1}) = \frac{e_{k-1} v_k}{n^2} > \frac{e_{k-2} v_{k-1}}{n^2},$$

the result follows from Theorem 4.9 □

.

Later we will need the following result.

Proposition 4.13. *Let (f) be a branch with semigroup of values $G_f = \langle v_0, \dots, v_g \rangle$. If (h) is a branch such that $d_X(f, h) < \frac{e_{k-1} v_k}{n^2}$, for some $0 < k \leq g$, then $\mathrm{I}(f, h) \in \langle v_0, \dots, v_{k-1} \rangle$.*

Proof. Let f_{k-1} be a $(k-1)$-th key-polynomial of f, so $\mathrm{I}(f_{k-1}, X) = \frac{n}{e_{k-1}}$ and $\mathrm{I}(f, f_{k-1}) = v_k$.

From our assumption, we get

$$d_X(f, h) < \frac{e_{k-1} v_k}{n} = d_X(f, f_{k-1}).$$

Applying the STI condition to f, f_{k-1} and h, we have $d_X(f_{k-1}, h) = d_X(f, h)$. Hence

$$\mathrm{I}(f, h) = \frac{\mathrm{I}(f_{k-1}, h)}{n/e_{k-1}} \cdot n = e_{k-1} I(f_{k-1}, h) \in \langle v_0, \dots, v_{k-1} \rangle.$$

□

4.3 The Theorems of Merle and Granja

Using what we have already developed in this chapter we will prove a result on factorization of power series that have high contact with a given branch. This will be the content of Theorem 4.15 which is very close to Granja's theorem which in turn is a generalization of Merle's result on polar curves. Merle's Theorem was proved over the field of complex numbers, using the notion of contact among curves through Puiseux parametrizations, while Granja's result was proved using Hamburguer-Noether expansions more suited for the positive characteristic case. The strategy we will use is that of Garcia Barroso and Ploski, which makes use of the log-distance $d(f, h)$.

Lemma 4.14. *Let ϕ be an irreducible power series such that, for some $k > 0$, $d_X(f, \phi) = \frac{e_{k-1} v_k}{n^2}$. Then $\mathrm{I}(\phi, X) \equiv 0 \bmod \frac{n}{e_{k-1}}$ and $\mathrm{I}(f, \phi) \equiv 0 \bmod v_k$.*

Proof. If $k = 1$, our hypothesis implies that $\mathrm{I}(f,\phi) = v_1\,\mathrm{I}(\phi,X)$ and obviously $\mathrm{I}(\phi,X) \equiv 0 \bmod 1$ and $\mathrm{I}(f,\phi) \equiv 0 \bmod v_1$.

If $k > 1$, our hypothesis implies that $\mathrm{I}(f,\phi) = \frac{e_{k-1}v_k}{n}\mathrm{I}(\phi,X)$, so it suffices to check that $\mathrm{I}(\phi,X) \equiv 0 \bmod \frac{n}{e_{k-1}}$. Our hypothesis also implies that $d_X(f,\phi) = \frac{e_{k-1}v_k}{n^2} > \frac{e_{k-2}v_{k-1}}{n^2}$, hence by Lemma 4.7 $n' = \mathrm{I}(\phi,X) \equiv 0 \bmod \frac{n}{e_{k-1}}$. □

Theorem 4.15 (Merle-Granja's Factorization Theorem). *Let (f) be a plane branch with semigroup of values $G_f = \langle v_0,\dots,v_g\rangle$ and associated integers e_i, n_i, $i = 0,\dots,g$. Let k be such that $1 \le k \le g$. If $h \in K[[X,Y]]$ is such that*

a) $\mathrm{I}(h,X) < \frac{n}{e_k}$, *and*

b) $\mathrm{I}(f,h) = \sum_{i=1}^{k}(n_i-1)v_i$,

then there is a factorization $h = h_1\cdots h_k \in K[[X,Y]]$ such that for any irreducible factor ϕ of h_i one has

i) $\mathrm{I}(h_i,X) = \frac{n}{e_i} - \frac{n}{e_{i-1}}$, *for $i \in \{1,\dots,k\}$,*

ii) $d_X(f,\phi) = \frac{e_{i-1}v_i}{n^2}$,

iii) $\mathrm{I}(\phi,X) \equiv 0 \bmod \frac{n}{e_{i-1}}$.

Proof. Let $h \in K[[X,Y]]$ be such that the conditions (a) and (b) hold. Firstly we check the following two claims:

Claim 1. If ϕ is an irreducible factor of h, then $d_X(f,\phi) \le \frac{e_{k-1}v_k}{n^2}$.

Indeed, suppose that there exists an irreducible factor ϕ of h such that $d_X(f,\phi) > \frac{e_{k-1}v_k}{n^2}$. By Lemma 4.7 (ii) we get that $\mathrm{I}(\phi,X) \equiv 0 \bmod \frac{n}{e_k}$, which is a contradiction, since $0 < \mathrm{I}(\phi,X) \le \mathrm{I}(h,X) < \frac{n}{e_k}$.

Claim 2. There exists an irreducible factor, ϕ of h such that $d_X(f,\phi) = \frac{e_{k-1}v_k}{n^2}$.

Indeed, if $d_X(f,\phi) < \frac{e_{k-1}v_k}{n^2}$ for all irreducible factor ϕ of f, by Proposition 4.13 we have that $\mathrm{I}(f,\phi) \in \langle v_0,\dots,v_{k-1}\rangle$ for all ϕ, then

$$\mathrm{I}(f,h) = \sum_{\phi}\mathrm{I}(f,\phi) \in \langle v_0,\dots,v_{k-1}\rangle.$$

But, by Remark 2.24, we have that $\mathrm{I}(f,h) = \sum_{i=1}^{k}(n_i-1)v_i \not\equiv 0 \bmod e_{k-1}$, a contradiction.

Now, let us denote by h_k the product of all irreducible factors ϕ of h such that $d_X(f,\phi) = \frac{e_{k-1}v_k}{n^2}$. So, we get $h = h'h_k$ with $d_X(f,\psi) < \frac{e_{k-1}v_k}{n^2}$ for all irreducible factor ψ of h'. So, from Lemma 4.14 we get $\mathrm{I}(f,h_k) \equiv 0 \bmod v_k$ and using Proposition 4.13 one has $\mathrm{I}(h',f) \in \langle v_0,\dots,v_{k-1}\rangle$.

Now write $\mathrm{I}(f,h') = m_0v_0 + \cdots + m_{k-1}v_{k-1}$ whith $m_0 \geq 0$ and $0 \leq m_i \leq n_i - 1$ and $\mathrm{I}(f,h_k) = m_kv_k$, $m_k \geq 0$. From the hypothesis we have

$$\sum_{i=1}^{k}(n_i-1)v_i = \mathrm{I}(f,h) = \mathrm{I}(f,h') + \mathrm{I}(f,h_k) = m_0v_0 + \cdots + m_{k-1}v_{k-1} + m_kv_k,$$

thus by Remark 2.24 we must have $m_0 = 0$ and $m_i = n_i - 1$, for $i \in \{1,\ldots,k\}$, so $\mathrm{I}(f,h_k) = (n_k-1)v_k$ and

$$\mathrm{I}(h_k,X) = \sum_{\phi}\mathrm{I}(\phi,X) = \sum_{\phi\in I}\left(\frac{\mathrm{I}(f,\phi)}{\frac{e_{k-1}v_k}{n}}\right) = \frac{\mathrm{I}(f,h_k)}{\frac{e_{k-1}v_k}{n}} = \frac{(n_k-1)v_k}{\frac{e_{k-1}v_k}{n}} = \frac{n}{e_k} - \frac{n}{e_{k-1}}.$$

We conclude our proof by induction on k. If $k = 1$, then $h = h'h_1$ with $\mathrm{I}(f,h') = m_0v_0 = 0$, hence h' is a unit. Call $h'h_1$ just h_1.

Suppose that Theorem 4.15 is true for $k-1$. Then we have that $h' = \frac{h}{h_k}$ satifies the assumptions of the theorem for $k-1$, so we may apply the inductive hypothesis and we are done.

The proof of (iii) follows from Lemma 4.7 (ii). □

To prove the following result we will recall a classical formula due essentially to Dedekind. We will denote by f_X and f_Y the partial derivatives of f.

Theorem 4.16 (Dedekind's Formula). *Let (f) be a plane branch defined over an algebraically closed field K and let $n = \mathrm{I}(f,X)$. Then one has*

$$\mathrm{I}(f,f_Y) \geq c + n - 1,$$

with equality if and only if $\mathrm{char}(K) \nmid n$.

Proof. For a detailed proof of this result see [RJ, Corollary 3.1.8]. □

Corollary 4.17 (Merle's factorization theorem). *Let (f) be a plane branch defined over an algebraically closed field K. Suppose that $G_f = \langle v_0,\ldots,v_g\rangle$, with $v_0 = n > 1$ and $n \not\equiv 0 \bmod \mathrm{char}(K)$. Then $f_Y = h_1\cdots h_g$ in $K[[X,Y]]$, where*

i) $\mathrm{I}(h_i,X) = \frac{n}{e_i} - \frac{n}{e_{i-1}}$, *for $i \in \{1,\ldots,g\}$, and*

ii) *if $\phi \in K[[X,Y]]$ is an irreducible factor of h_i, $i \in \{1,\ldots,k\}$,*

then

a) $d_X(f,\phi) = \dfrac{e_{i-1}v_i}{n^2}$;

b) $\mathrm{I}(\phi,X) \equiv 0 \bmod \dfrac{n}{e_{i-1}}$.

Proof. Since $n \not\equiv 0 \bmod \mathrm{char}(K)$ we have $\mathrm{I}(f_Y,X) = n-1$. By Dedekind's Formula (Theorem 4.16), we have $\mathrm{I}(f_Y,f) = n-1+c$ and from the Conductor Formula (Proposition 2.14), we have $c = \sum_{i=1}^{g}(n_i-1)v_i - n + 1$, so $\mathrm{I}(f_Y,X) = \sum_{i=1}^{g}(n_i-1)v_i$, then we can apply Theorem 4.15 to $h = f_Y$ and $k = g$ and we obtain the result. □

4.4 The Milnor number and the main result

Let K be an algebraically closed field of characteristic $p \geq 0$ and let $f \in K[[X, Y]]$ be an irreducible power series. The *Milnor number* of f is by definition

$$\mu(f) = \dim_K \frac{K[[X, Y]]}{\langle f_X, f_Y \rangle}.$$

The Milnor number is invariant under the action of an automorphism $\phi = (\phi_1(X, Y), \phi_2(X, Y))$ of $K[[X, Y]]$, as one may easily verify with the chain rule.

On the other hand, the Milnor number is not invariant by multiplication of f by a unit, as one can verify in the following example.

Example 4.18. *Suppose that* $\operatorname{char} K = p > 0$ *and let* $f = Y^p + X^{p+1} \in k[[X, Y]]$. *Then* (f) *is a plane branch such that*

$$\mu(f) = \infty, \qquad \text{but} \qquad \mu((1 + Y)f) = p^2 \neq \mu(f).$$

Deligne, in [De], has shown that in arbitrary characteristic, one has $\mu(f) \geq c$, where c is the conductor of G_f. It is classically known that if $\operatorname{char}(K) = 0$, then $\mu(f) = c$.

In what follows, we will give a necessary and sufficient condition, under the restrictive hypothesis that $p > \operatorname{mult}(f)$, to have $\mu(f)g = c$.

Since f is irreducible, then either f is regular in Y or in X of order $n = \operatorname{mult}(f)$. We may assume, after a change of coordinates in $K[[X, Y]]$, if necessary, that f is regular in Y of order $\operatorname{mult}(f)$, that is, $\operatorname{I}(f, X) = n$.

Theorem 4.19. *Suppose that* $p > \operatorname{mult}(f) = n$. *Then*

$$\operatorname{I}(f, f_Y) \leq \mu(f) + \operatorname{mult}(f) - 1$$

with equality if, and only if, $p \nmid v_k$, *for all* $k \in \{1, \ldots, g\}$.

Proof. Recall that we are assuming that f is irreducible and $\operatorname{I}(f, X) = \operatorname{mult}(f) = n$, hence f has as tangent cone Y^n. Since $p > \operatorname{mult}(f) = n$, the tangent cone of f_Y is Y^{n-1}, and consequently, any irreducible component ϕ of f_Y has as tangent cone Y^r for some $r \leq n - 1$, hence one has $\operatorname{I}(\phi, X) \leq n - 1$.

Now we state and prove a couple of claims.

Claim 1. For every irreducible factor ϕ of f_Y we have

$$\operatorname{I}(f_X, \phi) + \operatorname{mult}(\phi) \geq \operatorname{I}(f, \phi),$$

with equality if, and only if, $\operatorname{I}(f, \phi) \not\equiv 0 \bmod p$

Indeed, if $(X(t), Y(t))$ is a good parametrization of (ϕ), then

$$\text{mult}(X(t)) = \text{I}(X, \phi) = \text{mult}(\phi) \leq \text{mult}(f_Y) = n - 1 < n < p,$$

consequently, $\text{mult}(X(t)) \not\equiv 0 \mod p$, which implies

$$\text{mult}(X'(t)) = \text{mult}(X(t)) - 1.$$

Since ϕ is an irreducible factor of f_Y and $(X(t), Y(t))$ is a good parametrization of ϕ, we have

$$\frac{d}{dt} f(X(t), Y(t)) = f_X(X, Y)X'(t) + f_Y(X, Y)Y'(t) = f_X(X, Y))X'(t).$$

This implies that

$$\begin{aligned}\text{mult}(f(X(t), Y(t))) - 1 \leq \text{mult}\left(\frac{d}{dt} f(X(t), Y(t))\right) \\ &= \text{mult}(f_X(X(t), Y(t))X'(t)) \\ &= \text{mult}(f_X(X(t), Y(t))) + \text{mult}(X'(t)) \\ &= \text{mult}\,(f_X(X(t), Y(t))) + \text{mult}(X(t)) - 1,\end{aligned}$$

with equality if, and only if, $\text{I}(f, \phi) = \text{mult}(f(X(t), Y(t))) \not\equiv 0 \mod p$.

Therefore,

$$I(f_X, \phi) + \text{I}(X, \phi) \geq \text{I}(f, \phi),$$

with equality if and only if $\text{I}(f, \phi) \not\equiv 0 \mod p$, proving Claim 1.

Claim 2. Suppose that $p > \text{mult}(f)$ and let $f_Y = h_1 \cdots h_g$ be the Merle factorization of f_Y. Let ϕ be an irreducible factor of h_k. Then $\text{I}(f, \phi) \not\equiv 0 \bmod p$ if, and only if $v_k \not\equiv 0 \bmod p$.

Indeed, by Merle's Factorization Theorem (b), we can write $\text{mult}(\phi) = m_k \frac{n}{e_{k-1}}$, where $m_k \geq 1$ is an integer. Since $\text{mult}(\phi) \leq \text{mult}(f_Y) = \text{mult}(f) - 1 < p$, it follows that $\text{mult}(\phi) \not\equiv 0 \bmod p$, which implies that $m_k \not\equiv 0 \bmod p$.

By Merle's Factorization Theorem (a) we have

$$\text{I}(f, \phi) = \frac{e_{k-1} v_k}{n} \text{ mult}(\phi) = \frac{e_{k-1} v_k}{n} \cdot m_k \frac{n}{e_{k-1}} = m_k v_k$$

Therefore $\text{I}(f, \phi) \not\equiv 0 \bmod p$ if and only if $v_k \not\equiv 0 \mod p$.

Now we proceed with the proof of the theorem. Let $\mathcal{F}$ be the set of all irreducible factors of f_Y. Then by Claim 1,

$$\begin{aligned}\text{I}(f, f_Y) &= \textstyle\sum_{\phi \in \mathcal{F}} \text{I}(f, \phi) \leq \sum_{\phi \in \mathcal{F}} \text{I}(f_X, \phi) + \sum_{\phi \in \mathcal{F}} \text{mult}(\phi) = \mu(f) + \text{mult}(f_Y) \\ &= \mu(f) + \text{mult}(f) - 1.\end{aligned}$$

with equality if and only if $\text{I}(f, \phi) \not\equiv 0 \mod p$ for all $\phi \in \mathcal{F}$. According to Claim 2, $\text{I}(f, \phi) \not\equiv 0 \bmod p$ for all $\phi \in \mathcal{F}$ if and only if $v_k \not\equiv 0 \bmod p$ for $k \in \{1, \ldots, g\}$ and the theorem follows. □

Corollary 4.20. *Let* $f \in K[[X,Y]]$ *be an irreducible power series with* $\mathrm{mult}(f) = n > 1$ *and let* $n = v_0, \dots, v_g$ *be the minimal system of generators of* G_f*. Suppose that* $p = \mathrm{char}(K) > \mathrm{mult}(f)$*, then the following conditions are equivalent:*

i) $v_k \not\equiv 0 \mod p$*, for all* $k \in \{1, \dots, g\}$*;*

ii) $\mu(f) = c(f)$*.*

Proof. By the Dedekind's formula and Theorem 4.19 one has

$$c + n - 1 \leq \mathrm{I}(f, f_Y) \leq \mu(f) + n - 1,$$

with equality of the ends if and only if $p \nmid v_0 \cdots v_g$. □

Remarks.

1) Theorem 4.19 gives Deligne's inequality $\mu(f) \geq c$, under the hypothesis $p > \mathrm{mult}(f)$.

2) In [HRS], Hefez, Rodrigues and Salomão proved, without any assumption on the characteristic p, the following implication:

If $p \nmid v_0 \cdots v_g$*, then* $\mu(f) = c$*.*

The above result was stated as a conjecture in [GB-P2].

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